Minimal abductive solutions with explicit justification

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Abstract

Abductive problems and their solutions are presented by means of justification logic. We introduce additional meta-constructions in order to generate and compare different solutions to the same abductive problem. Our approach has three advantages: (i) it makes structurally explicit the solution to an abductive problem (as it has a syntactic nature); (ii) it gives a precise meaning to the notion of evidence; (iii) it provides clear definitions and procedures for the comparison of solutions that can be adapted to different needs.

Keywords: abduction, justification logic, evidence, explanation, minimality

1 Introduction

Although not every formal model of abduction deals explicitly with the concept of evidence/justification, many conceptual models of abduction (mainly *AKM* [1, 13, 26, 30, 34], *GW* [16] and the *eco-cognitive approach* [31–33]) address the role that these notions have for abductive reasoning.[1] For example, abductive reasoning can be seen as an inference whose input includes evidence, because it is '*thinking from evidence to explanation, a type of reasoning characteristic of many different situations with incomplete information.*' [1, p. 28]. This statement is compatible with the definition found in [16], for whom '*...abduction is a process of justifying an assumption, hypothesis or conjecture for its role in producing something in which the abducer has declared an interest*' (p. 40). That is, the evidence from the input allows for the justification of a given piece of information, relevant to the theory at hand. Nevertheless, as '[t]*he logic of abduction investigates the business of reasoning well in the absence of evidence*' [16, p. 100], new evidence is needed in order justify this information.[2]

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[1] Acronyms '*AKM*' and '*GW*' result from putting together the initials of surnames belonging to authors of whom are said to adopt a particular stance on abduction. For example, *GW* stands for Dov Gabbay and John Woods, who take an agent-based perspective on the matter.

[2] For more about a general perspective on abduction, seen in terms of *inputs* and *outputs*, refer to [10].

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Our approach to abduction combines these ideas concerning evidence and justification. In other words, given the absence of evidence in a theory Γ for a given piece of information, we define an abductive inference as the attainment of new evidence for Γ, so that the expanded theory is able to justify said information.

Formally, abstract models developing conceptual perspectives of abduction are based on a diversity of logical systems, like semantic tableaux [1], neighborhood semantics [29], epistemic logics [40, 44], dynamic logics [39], category theory [43], adaptive logics [7, 17, 35], paraconsistent logics [8], algorithmic-based logics [13] and many others. Each of these models emphasizes on particular aspects of abduction. In our proposal, as we aspire to give evidence and justification a central role, and following Gabbay's advice concerning the use of labeled formulas for both structuring sets of information and defining mechanisms of abductive inference [14, 15], we use the framework of justification logics [3, 4, 11].

We start by expressing Abductive Problems (*AP*s) and Abductive Solutions (*ASol*s) in justification logic. *ASol*s will not simply be formulas from which an unexplained formula can be derived, but formulas plus a *justification term* that provides a structurally explicit explanation of an *AP*. As additional gain, we provide a formal meaning for the notion of *evidence*, which captures some (though not all) of the intuitions behind this word.

The further layer of structure provided by justification terms and their operations enables us not only to build different *ASol*s, but also to compare them and evaluate them. A key notion for the evaluation is *minimality*: a minimal *ASol* is built exclusively from essential pieces of evidence for the problem at hand. Further, this notion will be of use to model a framework in which *ASol*s can be generated, classified and evaluated, according to the relation between their constituting evidence.

The outline of the paper is as follows: in Section 2 we give a brief summary of justification logic for the uninitiated. In Section 3 we express '*AP*' and '*ASol*' in justification logic and define some properties of *ASol*s. In Section 4 we advance a proper notion of minimality. In Section 5 we introduce the notion of a *family* of *ASol*s. A method for generating families of *ASol*s ('breeding') and assessing them according to suitable criteria is presented here. Finally, in Section 6 we discuss some avenues for future work.

2 Justification logic

This section recalls the basic system of justification logic, J_0, as well as the different systems J_{CS} that arise when J_0 is extended with additional axioms. All definitions and results up to (and including) Theorem 2.5 are standard and can be found in the source papers [3, 4, 27, 41].

Through the text, let *PAtm* be an enumerable set of atomic propositions.

DEFINITION 2.1

(*Justification terms*) Let C be an enumerable set of *constant justification terms* (*constants*, for shorthand) and V an enumerable set of *variable justification terms* (*variables*, for shorthand). Using Backus-Naur form notation (BNF notation, for short), *justification terms* t, u are given by

$$t, u ::= c \mid x \mid [t * u] \mid [t + u]$$

for $c \in C$ and $x \in V$. The square brackets in terms $[t * u]$ and $[t + u]$ will be omitted when no confusion arises; the set of all justification terms based on C and V is denoted by $\mathcal{T}$. A justification term t is said to be *atomic* if and only if $t \in C$ or $t \in V$, and it is said to be *composite* otherwise. Finally, for $t \in \mathcal{T}$, the set $Sub(t) \subseteq \mathcal{T}$ contains all composite and atomic terms constituting

t (so s is a *sub-term* of t if and only if $s \in Sub(t)$), and the set $Atm(t) \subseteq (C \cup V)$ contains all atomic terms constituting t (so s is an *atom* of t if and only if $s \in Atm(t)$).

DEFINITION 2.2
(Language $\mathcal{L}_J$) Formulas ϕ, ψ of the justification logic language $\mathcal{L}_J$ are given by

$$\phi, \psi ::= p \mid \neg\phi \mid \phi \vee \psi \mid t : \phi$$

with $p \in PAtm$ and $t \in \mathcal{T}$. *Justification formulas*, those of the form $t : \phi$, are read as *'the term t is a justification for ϕ'*. Other propositional connectives, as $\wedge$, $\rightarrow$ and $\leftrightarrow$, are defined in the standard way.

DEFINITION 2.3
(System J_0) Let ϕ, ψ be formulas in $\mathcal{L}_J$; let t, s be terms in $\mathcal{T}$. The scheme system J_0 is defined by the following axioms and rules, with *A2* and *A3* also called the *Application* axiom and the *Sum* axioms, respectively.

(*A1*)	$\vdash \phi$ for ϕ any $\mathcal{L}_J$-instance of *TAUT*, with *TAUT* a finite, sound and complete axiom scheme for propositional logic.
(*A2*)	$\vdash s : (\phi \rightarrow \psi) \rightarrow (t : \phi \rightarrow [s * t] : \psi)$.
(*A3*)	$\vdash (s : \psi \rightarrow [s + t] : \psi), \vdash (s : \psi \rightarrow [t + s] : \psi)$.
(*R1*)	From $\vdash \phi \rightarrow \psi$ and $\vdash \phi$ derive $\vdash \psi$.

DEFINITION 2.4
(Systems J_{CS}) Let CS be an arbitrary set of formulas in $\mathcal{L}_J$ of the form $c_n : c_{n-1} : \ldots : c_1 : \phi$, for $n \in \mathbb{N}$, $n \geq 1$, $\phi \in TAUT$ and $c_j \in C$ $(1 \leq j \leq n)$. In other words, let CS be such that

$$CS \subseteq \{c_n : c_{n-1} : \ldots : c_1 : \phi \mid n \geq 1, c_j \in C, 1 \leq j \leq n \text{ and } \phi \in TAUT\}.$$

Moreover, assume each such CS is 'closed under justifications', i.e.

$$c_n : c_{n-1} : \ldots : c_1 : \phi \in CS \text{ implies } c_{n+1} : c_n : c_{n-1} : \ldots : c_1 : \phi \in CS.$$

The system J_{CS} extends J_0 by taking all formulas in CS as axioms, i.e.

$$\mathsf{J}_{CS} := \mathsf{J}_0 + CS.$$

A set CS might have additional properties. For example,

- A set CS is *axiomatically appropriate* if and only if it contains a justification formula for each scheme in *TAUT*, i.e. if and only if

$$\phi \in TAUT \text{ implies } c_n : c_{n-1} : \ldots : c_1 : \phi \in CS, \text{ for } c_j \in C, 1 \leq j \leq n.$$

- A set CS is *total* if and only if it contains a justification formula for each scheme in *TAUT* and every string of constants in C, i.e. if and only if

$$\{c_n : c_{n-1} : \ldots : c_1 : \phi \mid c_j \in C, 1 \leq j \leq n \text{ and } \phi \in TAUT\} \subseteq CS.$$

For the sake of simplicity, strings of constants are usually replaced by single constants, as we will do through this paper.

It should be noted that, if a *CS* set is axiomatically appropriate, its associated J_{CS} system satisfies a property called *internalization*:

$$\text{If} \vdash_{\mathsf{J}_{CS}} \phi, \text{then} \vdash_{\mathsf{J}_{CS}} t:\phi, \text{for some term } t.$$

As J_0 is the J_{CS} system with $CS = \emptyset$, and the only 'truly' justification formulas in it are instances of $A2$ and $A3$, we have $\nvdash_{\mathsf{J}_0} t:\phi$ for every formula of the form $t:\phi$. In other words, justified reasoning under J_0 alone is limited to conditional assumptions, as *'if $s:\phi$ and $t:(\phi \rightarrow \psi)$ were the case, we could infer $t * s:\psi$'*. Given J_0's inability for producing justification formulas, we will define abductive reasoning using J_{CS} systems. What kind of J_{CS} system(s) we will employ is not relevant at the moment.

Finally, we have the deduction theorem [4] for every justification system.

THEOREM 2.5
(Deduction theorem) For any J_{CS} system, and any $\Gamma \subseteq \mathcal{L}_J$ and $\phi, \psi \in \mathcal{L}_J$, we have that $\Gamma, \phi \vdash_{\mathsf{J}_{CS}} \psi \Leftrightarrow \Gamma \vdash_{\mathsf{J}_{CS}} \phi \rightarrow \psi$.

2.1 Basic properties of justification terms

This section is dedicated to the definition of some properties of justification terms. These properties are basic, in the sense that they are definable through a simple constraint on the terms' syntactic structure. Later sections will use these properties in order to define more complex ones.

Let *Sub* and *Atm* be as in Definition 2.1.

DEFINITION 2.6
(Tracking sub-terms and atoms under operations) A refinement of *Sub* lets us track sub-terms occurring under specific operations. First, the function $Sub^* : \mathcal{T} \rightarrow \wp(\mathcal{T})$ receives a term, returning only its sub-terms occurring under application ('$*$'):

$$Sub^*(x) := \emptyset, \qquad Sub^*([s * t]) := Sub^*(s) \cup Sub^*(t) \cup \{s, t\},$$
$$Sub^*(c) := \emptyset, \qquad Sub^*([s + t]) := Sub^*(s) \cup Sub^*(t),$$

for $x \in V$ and $c \in C$. The function $Sub^+ : \mathcal{T} \rightarrow \wp(\mathcal{T})$, receiving a term and returning only its sub-terms occurring under sum ('$+$'), is defined analogously.

We also define Atm^* and Atm^+, for any $t \in \mathcal{T}$, in the following way:

$$Atm^*(t) := Sub^*(t) \cap (V \cup C), \qquad Atm^+(t) := Sub^+(t) \cap (V \cup C).$$

Thus, while Atm^* is the set of atomic sub-terms of t occurring under application, Atm^+ is the set of atomic sub-terms of t occurring under sum.

DEFINITION 2.7
(Operability function) The function $Op : \mathcal{T} \rightarrow \mathbb{N}$, receiving a justification term and returning a natural number understood as the term's *operability*, is recursively defined as follows:

$$Op(x) := 0, \qquad Op([s * t]) := Op(s) + Op(t) + 2,$$
$$Op(c) := 0, \qquad Op([s + t]) := Op(s) + Op(t) + 1,$$

with '$+$' the standard addition operation (different from $\mathbf{+}$, the one used for sums of justification terms.

Intuitively, the operability of a term stands for its complexity, and it is determined by the operations that put its sub-terms together. An application '$*$' is considered more complex than a sum '$+$' because, while inferring that a formula is justified by a term $[r * s]$ requires two uses of modus ponens rule (see axiom *A2*), inferring that a formula is justified by a term $[r + s]$ requires only one (see axiom *A3*). This suggests that building terms through applications should be considered as more difficult than building them through sum [5].

3 Abductive problem and abductive solution

We define abductive reasoning as an inferential process whose input is an *abductive problem* and whose output is an *abductive solution*. Let Γ be a set of $\mathcal{L}_J$ formulas closed under some system J_{CS}.[3] Intuitively, an abductive problem in Γ is a formula ψ of Γ for which the set cannot provide a justification. Naturally, an abductive solution for this problem is a justification of ψ with some term t, such that $t:\psi$ is a formula of an extended version of Γ, containing all the necessary formulas to derive $t:\psi$.[4] In formal terms:

DEFINITION 3.1 (Abductive problem and abductive solution).
Let Γ be a set of $\mathcal{L}_J$-formulas closed under J_{CS}.[5] A formula $\psi \in \Gamma$ is an *abductive problem* (*AP*) with respect to Γ if and only if $s:\psi \notin \Gamma$ for all $s \in \mathcal{T}$.

For defining an abductive solution, let Δ be a set containing justification formulas in $\mathcal{L}_J$ that allow the inference of a formula of the form $t:\psi$ from Γ, for some $t \in \mathcal{T}$. More precisely, let $\Delta \subseteq \{r:\phi \in \mathcal{L}_J \mid r \in \mathcal{T}, \phi \in \mathcal{L}_J\}$ be such that $\Delta \cap \Gamma = \emptyset$. A formula $t:\psi$ is an *abductive solution* (*ASol*) for the abductive problem $\psi \in \Gamma$ if and only if $t:\psi \in (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$. In other words, a formula $t:\psi$ is an abductive solution for the problem $\psi \in \Gamma$ if and only if, when extended with some Δ as specified above, the original set Γ plus Δ can derive $t:\psi$, for some $t \in \mathcal{T}$.

Note how, here, an abductive solution is the abductive problem plus a justification term for it, $t:\psi$.[6] This still allows the existence of multiple abductive solutions, as they might arise from different Δs. Even a fixed Δ can lead to multiple justifications for the same formula, as the following example shows.

EXAMPLE 3.2
Take a Γ such that $\{q, x:(p \to q)\} \subseteq \Gamma$ (with $x \in V$), and yet $t:q \notin \Gamma$ for every $t \in \mathcal{T}$. Thus, q is an *AP* with respect to Γ.

Consider now the set $\Delta = \{y:(o \to p), z:o\}$ with $y, z \in V$. From it, one can find the abductive solution $[x * [y * z]]:q$ (as this formula can be derived from $\Gamma \cup \Delta$ under J_{CS} via successive applications of *A2*). Still, more *ASol*s can be made available. Supposing that we have the additional axiom $c:((o \to p) \to ((p \to q) \to (o \to q)))$ in *CS*, we also get the slightly different *ASol* $[[[c * y] * x] * z]:q$, which includes a sub-term $[[c * y] * x]$ justifying a reasoning involving o and q,

[3] Unless specified otherwise, every $\Gamma \subseteq \mathcal{L}_J$ will be assumed to be closed under J_{CS}.

[4] At first sight, it might look odd that the abductive problem ψ belongs to the set Γ. The crucial fact here is that, even though Γ accounts for ψ, it does not provide a *justification* for it. In epistemic terms, ψ is an agent's abductive problem when she knows that ψ holds ($\psi \in \Gamma$), and yet she does not know *why* it does so ($s:\psi \notin \Gamma$ for every $s \in \mathcal{T}$).

[5] For a set $\Gamma \subseteq \mathcal{L}_J$, its closure under a given J_{CS} system will be denoted by $\Gamma^{\mathsf{J}_{CS}}$.

[6] Just like the definition for *AP*, there are other options. An appealing one is to consider the sets Δ as the abductive solutions themselves.

dropping the justification $[y * z]$ of p in exchange. Furthermore, we also obtain an infinite amount of redundant *ASol*s like $[s+[x*[y*z]]]:q$ for arbitrary $s \in \mathcal{T}$, although they do not represent a practical solution to the problem at hand.

As the example shows, each set Δ induces to a class of *ASol*s and so, each constructed *ASol* represents a certain way of making use of new evidence to justify a formula. The formulas in Δ can be used to build potentially 'better' *ASol*s, depending on the background information (formulas in Γ) and inferential powers (formulas in *CS*) accessible.[7] Consequently, the attainment of an *ASol* could greatly hinge on the known data and the way it is put together, no matter how instructive the new evidence from its respective Δ is.

Later on we will show how obtained *ASol*s can be constrained and filtered through several conditions, with some of them defined here below.

3.1 Basic properties of abductive solutions

What follows is the definition of properties usually desired for nontrivial *ASol*s to have, with the last one exclusive to our format of *ASol* as a justification formula. Let ψ be an *AP* in some set Γ, with $t:\psi \in (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$ one of its *ASol*s.

- *Relevance*: $t:\psi \notin \Delta^{\mathsf{J}_{CS}}$. Intuitively, a relevant *ASol* only complements the set in which the *AP* triggered, and thus, the derivation of $t:\psi$ must rely on Γ too.
- *Consistency*: $\bot \notin (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$. Intuitively, a consistent *ASol* cannot be derived by contradiction.
- *Non-vacuousness*. Take $u:\pi \in \Gamma$. Let $Pre^{\Gamma}_{u:\pi}$ be a set $Pre^{\Gamma}_{u:\pi} \subseteq \{r:\gamma \in \mathcal{L}_J \mid r:\gamma \in \Gamma\}$ (so $Pre^{\Gamma}_{u:\pi} \subseteq \Gamma$) satisfying
  1. $u:\pi \notin Pre^{\Gamma}_{u:\pi}$ (it does not contain $u:\pi$),
  2. $u:\pi \in (Pre^{\Gamma}_{u:\pi})^{\mathsf{J}_{CS}}$ (it derives $u:\pi$) and
  3. $\Lambda \subset Pre^{\Gamma}_{u:\pi}$ implies $u:\pi \notin \Lambda^{\mathsf{J}_{CS}}$ (no strict subset of $Pre^{\Gamma}_{u:\pi}$ derives $u:\pi$).

  In other words, given $u:\pi \in \Gamma$, a set $Pre^{\Gamma}_{u:\pi}$ contains justification formulas of Γ (different from the initial $u:\pi$) that are essential for deriving $u:\pi$ in Γ. Note how, from $Pre^{\Gamma}_{u:\pi} \subseteq \Gamma$ and the monotonicity of the derivation system, it follows not only that $(Pre^{\Gamma}_{u:\pi})^{\mathsf{J}_{CS}} \subseteq \Gamma^{\mathsf{J}_{CS}}$, but also that $Pre^{\Gamma}_{u:\pi} \subseteq (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$.

  We say that $t:\psi$ is a non-vacuous *ASol* in $(\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$ if and only if for every $s \in (Sub(t) \setminus \{t\})$ there is a formula $s:\phi \in Pre^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}}_{t:\psi}$, for some $\phi \in \mathcal{L}_J$ and some $Pre^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}}_{t:\psi}$. Intuitively, $t:\psi$ is a non-vacuous *ASol* when all the proper subterms of its justification are actually used in its derivation.

From now on we will assume the satisfaction of these properties in order to deal with nontrivial *ASol*s only. Also, given that regular justification formulas (i.e. justification formulas that are not *ASol*s) could also contain vacuous sub-terms, we will assume *non-vacuousness* for every justification formula. Finally, even if the properties presented below are intended to be satisfied by *ASol*s, we will sometimes assume, for the sake of convenience, its satisfaction by regular justification formulas. This, of course, will be made explicit.

[7] An interesting remark lies in the fact that a rich system of reasoning could make up for the lack of background information. In cases like these, a set Δ could even contain justified propositions only, being able to obtain various data concerning their relation through an extensive *CS* set. The opposite case is also possible: the lack of axioms could be meliorated by an exhaustive search of formulas to expand the background information, and use them to *abduce* a decent reasoning system.

4 Minimality

One of the main routes for delineating the concept of abduction is through its characterization as an argument and/or as an inference to the best explanation (IBE) [2, pp. 222–223]. Still, the identification of abduction with IBE is an open discussion. Some authors regard abduction only as an inferential activity of previously unknown information, leaving aside the subsequent selection of explanations [21, pp. 507–508; 23][8] ; some others claim a hardwired connection between IBE and abduction [12, 24, 26, 37].[9]

Without siding with any party on this debate, we will simply assume that abduction and IBE are not totally unrelated and our proposal is compatible with the view that IBE can be a result of abduction. In other words, once a set of *ASol*s for an *AP* is obtained, it is common for abductive models to move into a phase for selecting a subset of these. Typically, an *ASol* is chosen provided that it satisfies the properties *a good solution* has [28, 42]. A solution is good, or even the best among many, depending on the properties considered for evaluation.

Through this section and the next one we will show that, due to representing *ASol*s as justification formulas, many methods for evaluating and narrowing down collections of them are made available. We take advantage of these methods and define (i) a property that we call 'minimality', with which we characterize *ASol*s as arrangements of essential premises capable of resolving *AP*s, and later in Section 5, (ii) propose a framework in which groups of minimal *ASol*s can be classified, compared and selected under diverse criteria.

4.1 The elusiveness of minimality

One of the most desired properties for an *ASol* to have is called *minimality*. An *ASol* satisfies this property when, compared to any other *ASol*, it is the smallest in some sense, e.g. it contains the least possible information, it is the easiest one to infer, etc.

Being many ways to define this property, one of the main obstacles we find when evaluating it is its elusiveness. Minimality is elusive because the same *ASol* can be minimal in some sense while not being minimal in another.[10] For [1], this problem is possibly related to the 'product–process' distinction: *ASol*s can be evaluated both as products (frequently, as arguments or explanations) and processes (the reasoning behind their inference) (p. 74), without being both evaluations and their results necessarily related.

Following this suggestion, we argue that part of said elusiveness is caused by the propositional representation of *ASol*s. More specifically, under a purely propositional representation, an *ASol* is depicted as a piece of information, hiding away the components regarding its inferential process. In turn, the evaluation phase for minimality becomes a fragmented task constituted by at least two sub-phases: one assessing *ASol*s as products, and the other assessing them as inferential processes.

In process-oriented models of abduction we find a strong emphasis on the relation between the abducer, the available data and her inferential mechanisms. To mention some of them, an abductive process in [39] is represented as the transition between the states of a propositional dynamic logic model, in which every state is ruled by a different logic, such that some of these logics could be

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[8]The argument is usually that the criteria for selection is extra-theoretical [1] or, at least, because there are types of abduction not to be considered explanations [16, 31].

[9]This, in virtue of the presence of enumerative induction in their intersection [20, 25], or the meaningful relation between the relevance of an abductive hypothesis and its level of truthfulness.

[10]The fact that there are many definitions for this property is what makes of minimality a matter of preference [1, p. 65], having to choose one definition over others.

useful for solving the *AP* in hand. More recently, [6] adopts a GW agent-based perspective on abduction, where two sharing-information agents represent an *AP* (proponent) and an incomplete theory (opponent) that, for its completion, proponent builds an hypothesis through her dialogue with an opponent.[11]

Even if justification logics lack a dynamic nature and the language $\mathcal{L}_J$ does not include actions, its terms record the employed data and operations needed in order to build more complex terms; i.e. the materials necessary to execute an inference. In this sense, justification terms offer clues about the inferential process that takes place during the construction of justification formulas.

More specifically, our *ASol*s are evidence-based statements that find their formal expression in formulas of the form $t:\psi$. Thus, contrary to their purely propositional counterparts, these *ASol*s represent a product ($t:\psi$, a justification formula) and, in a syntactic picture, the process's blueprint (the ordered J_{CS}-combination of its premises codified in t). This representation of *ASol*s allows for assessing minimality in different ways including, not only product-based methods of evaluation, but also methods focused on the evaluation of their premisses' structure.

With this idea in mind we will review, in the subsection below, three definitions of minimality corresponding to the most representative notions of what is usually understood as a minimal *ASol*. Finally, in virtue of the advantages it offers over the others under our approach to abductive reasoning, we will choose one of these characterizations of minimality, in order to fully develop it through the rest of this paper. We will show that this notion of minimality enables the evaluation of *ASol*s with respect to many aspects, most of which remain implicit in a propositional representation of abduction.

4.2 Three notions of minimality

Typically, there are three senses in which an *ASol* is considered to be minimal.

(i) It is *a proper subset of the explanation at hand* [1, p. 71].[12] Under this view, *ASol*s are taken as products of argument construction: a minimal *ASol* is the smallest argument that can be build with the available information.

(ii) It is *informationally economic* ([1, p. 181], [2, p. 228]). An *ASol* is minimal in this sense, provided that it triggers the least number of changes in the initial theory or database.

(iii) It is *inferentially economic*.[13] This notion is focused on the process for deriving *ASol*s, more than on the result of this process. From a logical point of view, economy of an *ASol* is determined by the inferential resources needed in order to derive it.

Definition (ii) is an interesting candidate, as our format of *ASol* allows for at least two ways of defining it. Given an *AP* $\psi \in \Gamma$ and an *ASol* $t:\psi \in (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$, the formula $t:\psi$ is minimal provided that the extension of Γ with Δ produces the least number of new logical consequences in Γ, with respect to any other Δ' corresponding to a different *ASol* for ψ. Alternatively, $t:\psi$ is minimal provided that, apart from ψ, and compared to any other term constituting a different *ASol* for ψ, the term t justifies the least number of formulas in $(\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$.[14] As of (iii), a natural option for

[11] Other approaches to abduction, based on different frameworks of dialogical nature, like interrogative models [22] and abstract argumentation [9, 18, 19], are also interesting candidates for formalizing abductive processes.

[12] For [1], an *ASol* is an explanation of the formula triggering the *AP*.

[13] From an algorithmic perspective, this notion of minimality is concerned with the computational cost of deriving *ASol*s (see [13]); from an agent's perspective, inferential economy is understood as *cognitive economy* [16].

[14] There is no restriction regarding terms justifying multiple formulas, and this is the reason why an *ASol* is not necessarily restricted to justify a single formula.

its definition is using the *Op* function: an inferentially economical *ASol* is the one with the minimum *Op* value.

However, an immediate disadvantage of these notions is that they cannot deal with uninteresting *ASol*s. In the case of (ii), it would be easy to have an *ASol* to be minimal in virtue of having no consequences in the initial set of formulas. In the case of (iii), because operability of sums ($+$) is lower than operability of applications ($*$) (see Definition 2.7), it would be very possible to have minimal *ASol*s of the form $a+b:\psi$, containing irrelevant premises by the nature of sum operation.

This observation leaves us with (i).[15] In our approach, an *ASol* $t:\psi$ for an *AP* $\psi \in \Gamma$ is minimal in this sense when its premises cannot be rearranged into a smaller argument, represented by a justification formula $s:\psi$ that could be potentially built by the J_{CS}-combination of the premises constituting $t:\psi$. Below we fully define this notion of minimality, and show that *ASol*s that are minimal in this sense are nontrivial *ASol*s constituted by the minimum number of premises, necessary to confront the given *AP*. Later, in Section 5, we present an interesting framework of generation, classification and evaluation of *ASol*s, which can be defined by the aid of this notion of minimality.

4.3 Sub-minimality

In order to proceed we provide a formal definition of minimality in the sense of (i), and show its application through an example. Finally, we prove that this property is constructive; an *ASol* is minimal if and only if its premises are minimal. In Section 5 we will show that this fact enables us to create new minimal *ASol*s by the J_{CS}-combination of available minimal *ASol*s. In order to avoid confusions, from now we will call this property '*Sub*-minimality'.

DEFINITION 4.1
(*Sub-minimality*) The formula $t:\psi$ is a *Sub*–minimal *ASol* of ψ in $(\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$ (with $t:\psi \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}} \subseteq (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$) if and only if for every other justification formula $s:\psi \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ it is not the case that $|Sub(s)| < |Sub(t)|$.

Given that $Sub^{+}(t) = \emptyset$ for every *Sub*-minimal *ASol* constituted by term t, it is easy to see that this property is useful for getting rid of redundant *ASol*s. The following example showcases more of the qualities that *Sub*-minimal *ASol*s have. Let $q \in \Gamma$ be an *AP*.

EXAMPLE 4.2
Consider the statements '*gunpowder residue was found on the crime scene* (p)'; '*a firearm was shot on the crime scene* (q)'; '*someone dropped the contents of an unused bullet in the floor of the crime scene* (o)'. Also, let evidence x represent a bag of gunpowder residues found on a crime scene, and y, a common sense reasoning justifying $(p \to q)$.

Now suppose $\{q, x:o, x:p, y:(p \to q), [y*x]:q\} \subseteq (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$ (with $x, y \in V$ and $y:(p \to q) \in \Delta$), such that $y*x:q$ is an *ASol* for q in this set, generated by the combined union of Γ and a set Δ expanding the existing data in Γ.[16] It is the case that $y*x:q$ is a *Sub*-minimal *ASol*, as any other different combination of $x:p, y:(p \to q) \in Pre_{[y*x]:q}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}}$ under the system results in an unnecessarily

[15] Of course, combining these notions of minimality could lead to interesting conditions for *ASol*s to satisfy. However, for reasons of space we will not address these here.

[16] It is intuitive to think that a bag of gunpowder residues (x) can be used as a piece of evidence to justify more than one statement involving the existence of gunpowder, which is the case of propositions o and p.

longer justification for q, e.g. $[y * x] + y : q$. (Naturally, if $x, y \notin V$, we could not easily guarantee *Sub*-minimality for our *ASol*; we would have to corroborate the *Sub*-minimality of both premises.)

A finer distinction for *Sub*-minimality can be made. Suppose we also have $x : q \in \Gamma$ (with x the only available term for q in Γ), and a slightly different *AP*: q triggers as an *AP* in Γ because of *the absence of a composite justification for q*.[17] As long as we use the same premises as above, in this new scenario we have that our previous *ASol* $y * x : q$ is also *Sub*-minimal in $(\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$, the reason being that even if $x \in Sub(y * x)$, we have that $x : q \notin Pre^{(\Gamma \cup \Delta)^{\mathsf{J}_{CS}}}_{[y*x]:q}$. In other words, a *Sub*-minimal *ASol* is the smallest arrangement of a collection of premises, interpreting pieces of evidence in a specific manner: clearly, given $y : (p \to q) \in Pre^{(\Gamma \cup \Delta)^{\mathsf{J}_{CS}}}_{[y*x]:q}$ we have that, in $Pre^{(\Gamma \cup \Delta)^{\mathsf{J}_{CS}}}_{[y*x]:q}$, the term x is interpreted as a justification for p only.[18]

As we can see, *Sub*-minimality distinguishes between the (nontrivial) arrangement of the premises of an *ASol*, and the number of sub-terms constituting its term. More specifically, an *ASol* $u : \psi \in (\Gamma \cup \Delta')^{\mathsf{J}_{CS}}$ with $|Sub(u)| > |Sub(t)|$ can be *Sub*-minimal too, no matter if $t : \psi$ is also *Sub*-minimal. Consequently, a set of *Sub*-minimal *ASol*s for the same *AP* can be further evaluated under other notions of minimality. (For example, combining it with minimality in the sense of (iii), we could additionally filter *ASol*s with complex processes of inference involving numerous terms.)

The example above suggests something else: *Sub*-minimality of *ASol*s depends on the *Sub*–minimality of their components. This fact is a consequence of the constructive character between justification formulas and their properties, i.e. the properties of a justification formula are determined by the properties of its premises. In this sense, *Sub*-minimality is a constructive property, as it is shown in Theorem 4.5.

In order to prove Theorem 4.5 we first show some useful results.

PROPOSITION 4.3
Given the constructive character of justification terms, $s \in Sub(t)$ if and only if $Sub(s) \subseteq Sub(t)$, for every $s, t \in \mathcal{T}$.[19]

Next we show, for every justification term, the equality between being a proper sub-term s, and being a premise for which s is its term.

THEOREM 4.4
We have $s \in (Sub(t) \setminus \{t\})$ if and only if $s : \phi \in Pre^{\Gamma}_{t:\psi}$, for every $s, t \in \mathcal{T}$, every $s : \phi, t : \psi \in \mathcal{L}_J$, some $\phi \in \mathcal{L}_J$ and some set of formulas Γ.

PROOF. The *left to right* direction follows from non-vacuousness (see Subsection 3.1), which is assumed for every justification term. For the opposite direction suppose $s \notin (Sub(t) \setminus \{t\})$. We know that $Sub(s) \not\subset (Sub(t) \setminus \{t\})$ by Proposition 4.3, which means that $Pre^{\Gamma}_{t:\psi}$ does not contain justification formulas whose J_{CS}-combination has as result, for some $\phi \in \mathcal{L}_J$, a formula $s : \phi$, thus $s : \phi \notin Pre^{\Gamma}_{t:\psi}$. □

Now we are ready to demonstrate the constructive character of *Sub*-minimality property.

[17] Many other special cases of *AP* like this are definable in our approach.

[18] Under a more relaxed interpretation of evidence, where a detailed account explaining the use of a firearm in the crime scene is not needed, we could have a corresponding set of premises containing $x : q$ as one of its elements.

[19] The proof of this fact is easily obtained by considering that $r \in Sub(r)$ for every $r \in \mathcal{T}$.

THEOREM 4.5
Let $s:\phi$ be a formula in $\mathcal{L}_J$. An *ASol* $t:\psi \in (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$, with $t:\psi \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$, is *Sub*-minimal if and only if every $s:\phi \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ is *Sub*-minimal.

PROOF. From *left to right*, let $s:\phi \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ be non-*Sub*–minimal. Then, by definition of *Sub*-minimality (see Definition 4.1) there is a formula $r:\phi \in (Pre_{s:\phi}^{E^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ (with $E^{\mathsf{J}_{CS}} \subseteq (\Gamma\cup\Delta)^{\mathsf{J}_{CS}}$), such that $|Sub(r)| < |Sub(s)|$. Consequently, there is a $u:\psi \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ where u is built from elements of $Sub(t)$, except that $r \in Sub(u)$ but $s \notin Sub(u)$. Therefore, $|Sub(u)| < |Sub(t)|$, which implies that $t:\psi$ is not a *Sub*-minimal *ASol* for ψ.

For the opposite direction, assume that $t:\psi$ is not *Sub*-minimal. Then, there is a $u:\psi \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ with $|Sub(u)| < |Sub(t)|$. As a consequence, $\{r\} \subseteq (Sub(t) \setminus Sub(u))$ for some $Sub(r)$. By Proposition 4.3, $r \in Sub(t)$, and so, $r:\gamma \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ (for some $\gamma \in \mathcal{L}_J$) by Theorem 4.4.

For the last part of the proof for this direction we must show that $r:\gamma$ is indeed non-*Sub*–minimal. In order to do it we analyze two possible cases: $r:\gamma = t:\psi$ and $r:\gamma \neq t:\psi$.

1. If $r:\gamma = t:\psi$, then $r:\gamma$ is not *Sub*-minimal because, as supposed at the beginning of the proof, $t:\psi$ is not *Sub*-minimal.
2. If $r:\gamma \neq t:\psi$, then γ is essential for the justification of ψ. Given that u is also a justification of ψ, and 'smaller' than t, there should be a term $e \in Sub(r)$ such that $e:\gamma \in (Pre_{r:\gamma}^{A^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ and $e:\gamma \in Pre_{u:\psi}^{B^{\mathsf{J}_{CS}}}$, with $A^{\mathsf{J}_{CS}} \subseteq (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$ and $B^{\mathsf{J}_{CS}} \subseteq (\Gamma \cup \Delta)^{\mathsf{J}_{CS}}$. Naturally, it must be the case that $|Sub(e)| < |Sub(r)|$. Therefore, $r:\gamma$ is not *Sub*-minimal.

In conclusion, $r:\gamma \in (Pre_{t:\psi}^{(\Gamma\cup\Delta)^{\mathsf{J}_{CS}}})^{\mathsf{J}_{CS}}$ is indeed non-*Sub*–minimal. □

It has been shown that, in order to guarantee *Sub*-minimality of *ASol*s, their premises must be *Sub*-minimal. To put it in another way, given the constructive character of justification formulas, our *ASol*s represent a modular arrangement of evidence. This modular aspect clearly implies an advantage: non-*Sub*–minimal *ASol*s can be turned into *Sub*-minimal *ASol*s by carefully revising each of their premises, and upgrade or remove the faulty ones.

Further, *Sub*-minimality leads to an interesting framework of generation, classification and evaluation of *ASol*s. In virtue of the constructive character of this definition of minimality (see Theorem 4.5), combining premises of available *ASol*s for a given *AP* allows, in many cases, for the generation of new *Sub*-minimal *ASol*s for the same *AP*. Generated *Sub*-minimal *ASol*s, together with their generating *Sub*-minimal *ASol*s, can be classified and evaluated according to the premises they share. We will call *families of Sub-minimal ASols* to this framework and flesh it out below.

5 Families of *Sub*-minimal Abductive Solutions

The framework of families allows for a two-part method for generating, organizing and narrowing down collections of *Sub*-minimal *ASol*s. The first part is called *breeding*, consisting in the simultaneous generation and classification of *Sub*-minimal *ASol*s for a given *AP*. The second part consists in evaluating and selecting said *Sub*-minimal *ASol*s, with respect to the place they occupy in said classification, and their ability of adapting to the theory in turn (represented by a set of formulas).

There are three main advantages for working with the framework of families. (i) New sub-minimal *ASol*s, hidden in the combination of two other *Sub*-minimal *ASol*s, can be built. (ii) New properties of *Sub*-minimal *ASol*s can be made explicit by the structure under which they are classified. (iii) Therefore, new methods of evaluation for these *ASol*s are made available.

5.1 Breeding the best minimal explanation

This part and Section 5.2 shows how *Sub*-minimal *ASol*s can be simultaneously generated and classified within the framework of families. Later, in Section 5.3, we will talk about some of the methods of evaluation this framework enables, in order to select individual *Sub*-minimal *ASol*s, and suggest some ideas in Section 6 to extended these methods of evaluation to groups of *Sub*-minimal *ASol*s.

We begin by defining the smallest group of *Sub*-minimal *ASol*s existing in this framework, which is a family. Later, we will define broader groups: generations and offsprings. A generation is a set formed by the *Sub*-minimal *ASol*s generated by all the available families at a given point, and an offspring, the set formed by the union of these generations.

DEFINITION 5.1
(*Families*) Let $(\mathbf{AS})_{\psi\in\Gamma}$ be the set of currently available *Sub*-minimal *ASol*s for some *AP* $\psi \in \Gamma$. Given two different $s:\psi, t:\psi \in (\mathbf{AS})_{\psi\in\Gamma}$, a family formed by *family heads* $s:\psi$ and $t:\psi$ is a set $\mathcal{F}^{s,t}_{\psi\in\Gamma} \subseteq ((\mathbf{AS})_{\psi\in\Gamma})^{\mathsf{J}_{CS}}$ such that, apart from containing $s:\psi$ and $t:\psi$, it also contains all new *Sub*-minimal *ASol*s for the *AP* $\psi \in \Gamma$ that can be generated by the combination of their premises.

In other words, if $s:\psi \in (\Gamma \cup \Delta_1)^{\mathsf{J}_{CS}}$ and $t:\psi \in (\Gamma \cup \Delta_2)^{\mathsf{J}_{CS}}$, for some Δ_1 and Δ_2 with $(\Delta_1 \cup \Delta_2) \cap \Gamma = \emptyset$, then $\mathcal{F}^{s,t}_{\psi\in\Gamma} \subseteq (Pre^{(\Gamma\cup\Delta_1)^{\mathsf{J}_{CS}}}_{s:\psi} \cup Pre^{(\Gamma\cup\Delta_2)^{\mathsf{J}_{CS}}}_{t:\psi})^{\mathsf{J}_{CS}} \subseteq (\Gamma \cup \Delta_1 \cup \Delta_2)^{\mathsf{J}_{CS}}$ is defined as

$$\begin{aligned}\mathcal{F}^{s,t}_{\psi\in\Gamma} := \{s:\psi, t:\psi, u:\psi \in (Pre^{(\Gamma\cup\Delta_1\cup\Delta_2)^{\mathsf{J}_{CS}}}_{s:\psi,t:\psi})^{\mathsf{J}_{CS}} \mid \\ u:\psi \notin (\mathbf{AS})_{\psi\in\Gamma} \text{ is a } \textit{Sub}\text{-minimal } \textit{ASol}, \text{ for} \\ \psi \in \Gamma \text{and } s:\psi, t:\psi \in (\mathbf{AS})_{\psi\in\Gamma}\}\end{aligned}$$

with $(Pre^{(\Gamma\cup\Delta_1\cup\Delta_2)^{\mathsf{J}_{CS}}}_{s:\psi,t:\psi})^{\mathsf{J}_{CS}}$, the abbreviation for $(Pre^{(\Gamma\cup\Delta_1)^{\mathsf{J}_{CS}}}_{s:\psi} \cup Pre^{(\Gamma\cup\Delta_2)^{\mathsf{J}_{CS}}}_{t:\psi})^{\mathsf{J}_{CS}}$. (Note that a single *Sub*-minimal *ASol* can be head of different families.)

Naturally, not every pair of *Sub*-minimal *ASol*s can be combined to obtain a new *Sub*-minimal *ASol* for the same *AP*. For this reason, we will assume the condition $(\mathcal{F}^{s,t}_{\psi\in\Gamma} \setminus \{s:\psi, t:\psi\}) \neq \emptyset$ for every family $\mathcal{F}^{s,t}_{\psi\in\Gamma}$.

It should be mentioned that defining families as the combination of two, but no more justification formulas, leads to a finer grained classification of *ASol*s, thus obtaining manageable sets of combinable *ASol*s with the smallest cardinality as possible. Still, we must prove that every new *Sub*-minimal *ASol* that can be generated under the J_{CS}-combination of all available *Sub*-minimal *ASol*s, can be generated in the framework of families, albeit with some conditions, as it is shown below.

THEOREM 5.2
Let $(\mathbf{AS})_{\psi\in\Gamma}$ be the set of every currently available *Sub*-minimal *ASol* for the given *AP* $\psi \in \Gamma$. Provided that we accept families containing weak *ASols* for ψ, like $s:(t:\psi) \in \mathcal{L}_J$ (for $s,t \in \mathcal{T}$), and adding the possibility of using *CS* formulas for further combinations, every new *Sub*-minimal

ASol for $\psi \in \Gamma$ contained in $((\mathbf{AS})_{\psi\in\Gamma})^{\mathsf{J}_{CS}}$ (with total *CS*), can be generated in the framework of families.

PROOF. We know that $((\mathbf{AS})_{\psi\in\Gamma})^{\mathsf{J}_{CS}}$ contains all new *Sub*-minimal *ASol*s for the *AP* $\psi \in \Gamma$ that can be generated by the J_{CS}-combination of every set of premises constituting each element in $(\mathbf{AS})_{\psi\in\Gamma}$. Suppose there is a *Sub*-minimal new *ASol* $r:\psi \in ((\mathbf{AS})_{\psi\in\Gamma})^{\mathsf{J}_{CS}}$ (i.e. $r:\psi \notin (\mathbf{AS})_{\psi\in\Gamma}$). Further, suppose that $r:\psi \notin \mathcal{F}^{s,t}_{\psi\in\Gamma}$, for every pair of *Sub*-minimal *ASol*s $s:\psi, t:\psi \in ((\mathbf{AS})_{\psi\in\Gamma})^{\mathsf{J}_{CS}}$. This means that $r:\psi$ is constituted by the J_{CS}-combination of three or more sets of premises belonging to members of $(\mathbf{AS})_{\psi\in\Gamma}$.

Let $r:\psi \in (Pre^{(\mathbf{AS})_{\psi\in\Gamma}}_{1:\psi,\ldots,n:\psi})^{\mathsf{J}_{CS}}$, with $1:\psi,\ldots,n:\psi \in (\mathbf{AS})_{\psi\in\Gamma}$ and $n > 2$. In addition to $r:\psi$, the set $(Pre^{(\mathbf{AS})_{\psi\in\Gamma}}_{1:\psi,\ldots,n:\psi})^{\mathsf{J}_{CS}}$ contains other new *Sub*-minimal *ASol*s for the *AP* $\psi \in \Gamma$. In fact, for every $i:\psi \in (1:\psi,\ldots,n:\psi)$ there is another $j:\psi \in (1:\psi,\ldots,n:\psi)$, such that $u:\psi \in (Pre^{(\mathbf{AS})_{\psi\in\Gamma}}_{i:\psi,j:\psi})^{\mathsf{J}_{CS}}$, for some *Sub*-minimal *ASol* $u:\psi$, with $u:\psi \notin (\mathbf{AS})_{\psi\in\Gamma}$ and $u:\psi \in ((\mathbf{AS})_{\psi\in\Gamma})^{\mathsf{J}_{CS}}$. If this were not the case, there would be a $k:\psi \in (1:\psi,\ldots,n:\psi)$, constituted by premises incompatible with the set of premises of each member of $((1:\psi,\ldots,n:\psi)\setminus k:\psi)$.

But this cannot be possible, as it would render $k:\psi$ incompatible with this set of *Sub*-minimal *ASol*s and so, we would have $k:\psi \notin (1:\psi,\ldots,n:\psi)$ and consequently, $r:\psi \notin ((1:\psi,\ldots,n:\psi)\setminus k:\psi)^{\mathsf{J}_{CS}}$. Therefore, there must be a pair $l:\psi, m:\psi \subseteq ((\mathbf{AS})_{\psi\in\Gamma})^{\mathsf{J}_{CS}}$, generated by the combination of all members in $(1:\psi,\ldots,n:\psi)$, such that $r:\psi \in \mathcal{F}^{l,m}_{\psi\in\Gamma}$. In conclusion, every generable *Sub*-minimal *ASol* can be generated by a family given certain conditions (which we asume only through this proof). □

In order to maintain a simple formalism, from now on we will omit, for sets of premises, all super-indexes indicating their set of reference. Any potential confusion will be cleared out in virtue of the context.

DEFINITION 5.3
(*Close and distant relatives*) We call relatives to the members of the same family. As the premises bewteen relatives can have some or no relation at all, we distinguish between *close* and *distant* relatives. Let $(\mathbf{AS})_{\psi\in\Gamma}$ be, once again, the set containing every currently available *Sub*-minimal *ASol* for the *AP* $\psi \in \Gamma$. For any pair $s:\psi, t:\psi \in \mathcal{F}^{a,b}_{\psi\in\Gamma}$ of any family $\mathcal{F}^{a,b}_{\psi\in\Gamma}$, the formulas $s:\psi$ and $t:\psi$ are close relatives if and only if there is some *Sub*-minimal $r:\psi \in (Pre_{s:\psi,t:\psi})^{\mathsf{J}_{CS}}$, be it new $(r:\psi \notin (\mathbf{AS})_{\psi\in\Gamma})$ or not $(r:\psi \in (\mathbf{AS})_{\psi\in\Gamma})$ (i.e. at least one of their premises is interchangeable). Otherwise, $s:\psi$ and $t:\psi$ are distant relatives.

In the example below we depict a family and its generated members, who happen to be distant relatives. For reasons of space and convenience, a justification formula $t:\phi$ will be abbreviated as t^{ϕ}.

EXAMPLE 5.4
Given $x_1:(q\to o), y_1:(r\to o), o \in \Gamma$, with $o \in PAtm$ an *AP* in this set, let $\mathcal{F}^{u,s}_{o\in\Gamma}$ be constituted by family heads $u:o = [x_1 * [x_2 * [x_3 * x_4]]]:o$ and $s:o = [y_1 * [y_2 * y_3]]:o$ (with $x_n, y_n \in V$ for $n \in \mathbb{N}$), containing the premises shown below.

$$u:o = [x_1^{q\to o} * [x_2^{(r\to q)} * [x_3^{(p\to r)} * x_4^{p}]^{r}]^{q}]:o, \quad s:o = [y_1^{(r\to o)} * [y_2^{(q\to r)} * y_3^{q}]^{r}]:o$$

This pair of *ASol*s does constitute a family; the combination of their premises leads to new *Sub*-minimal *ASol*s (for the same reason, family heads are close relatives). More specifically,

$[y_1 * [x_3 * x_4]] : o, [x_1 * y_3] : o \in (Pre_{u\,:\,o,s\,:\,o})^{\mathsf{J}_{CS}}$, and so $[y_1 * [x_3 * x_4]] : o, [x_1 * y_3] : o \in \mathcal{F}^{u,s}_{o \in \Gamma}$.

$$[y_1^{(r \to o)} * [x_3^{(p \to r)} * x_4^{p}]^{r}] : o, \qquad [x_1^{q \to o} * y_3^{q}] : o$$

However, because the information justified by their sub-terms is incompatible, no premise can be interchanged between $[y_1 * [x_3 * x_4]] : o$ and $[x_1 * y_3] : o$. Consequently, these members of family $\mathcal{F}^{u,s}_{o \in \Gamma}$ are distant relatives.

Finally, it should be noted that the only *Sub*-minimal *ASol*s for the *AP* in question, generated by pairs of close relatives that are not family heads, are their common relatives. In other words, as it will be shown in the following theorem, families are not *incestuous*, i.e. (close) relatives that are not family heads cannot generate new *Sub*-minimal *ASol*s for the given *AP*.

THEOREM 5.5
Let $\psi \in \Gamma$ be an *AP*. For every family $\mathcal{F}^{s,t}_{\psi \in \Gamma}$, no combination of (close) relatives in $(\mathcal{F}^{s,t}_{\psi \in \Gamma} \backslash \{s : \psi, t : \psi\})$ generates new *sub*-minimal *ASol*s for ψ.

PROOF. Let $r : \psi \notin \mathcal{F}^{s,t}_{\psi \in \Gamma}$, but $r : \psi \in \mathcal{F}^{a,b}_{\psi \in \Gamma}$ with $a : \psi, b : \psi \in (\mathcal{F}^{s,t}_{\psi \in \Gamma} \backslash \{s : \psi, t : \psi\})$. Then, $Pre_{a\,:\,\psi}$ and/or $Pre_{b\,:\,\psi}$ contain, at least, one premise that is not contained in $Pre_{s\,:\,\psi}$ and $Pre_{t\,:\,\psi}$.

But this cannot be possible as, by definition of families, $a : \psi$ and $b : \psi$ are constituted by the J_{CS}-combination of $Pre_{s\,:\,\psi}$ and $Pre_{t\,:\,\psi}$ only. In conclusion, if $r : \psi \notin \mathcal{F}^{s,t}_{\psi \in \Gamma}$, then $r : \psi \notin \mathcal{F}^{a,b}_{\psi \in \Gamma}$ for every $a : \psi, b : \psi \in (\mathcal{F}^{s,t}_{\psi \in \Gamma} \backslash \{s : \psi, t : \psi\})$. □

We will now extend our framework defining groups of *Sub*-minimal *ASol*s of broader scope, in which families are their building blocks. We call these groups generations and offsprings.

DEFINITION 5.6
(*Generations* and *offsprings*) A generation is the set of all *Sub*-minimal *ASol*s for a given *AP*, generated by all available families at some point. In order to maintain an order, we will label every generation. If $(\mathbf{AS})_{\psi \in \Gamma}$ is the set of all currently available *Sub*-minimal *ASol*s for an *AP* $\psi \in \Gamma$, containing no new *Sub*–minimal *ASol*s generated by some family, then we will say that $(\mathbf{AS})_{\psi \in \Gamma}$ is the first generation of *ASol*s for $\psi \in \Gamma$; in symbols, $Gen^{1st}_{\psi \in \Gamma} = (\mathbf{AS})_{\psi \in \Gamma}$. Naturally, the second generation is the set containing all new *Sub*-minimal *ASol*s generated by every family whose family heads are elements of $(\mathbf{AS})_{\psi \in \Gamma}$, and so on.

Formally, let $(\mathbf{AS})^{m}_{\psi \in \Gamma}$ be the set of new *Sub*-minimal *ASol*s for ψ generated by the l^{th} iteration of families for this *AP*, with $l < m < n$. Then,

$$Gen^{n}_{\psi \in \Gamma} := \{t : \psi \in \mathcal{F}^{r,s}_{\psi \in \Gamma} \mid r : \psi, s : \psi \in (\mathbf{AS})^{m}_{\psi \in \Gamma} \text{but } t : \psi \notin (\mathbf{AS})^{m}_{\psi \in \Gamma}\}.$$

Naturally, $Gen^{n}_{\psi \in \Gamma} = (\mathbf{AS})^{n}_{\psi \in \Gamma}$. Further, we define the offspring of *Sub*-minimal *ASol*s for the *AP* $\psi \in \Gamma$ as the set formed by the union of every generation of *Sub*-minimal *ASol*s for this *AP*. More precisely, for $n \in \mathbb{N}$,

$$Off_{\psi \in \Gamma} := \{Gen^{1}_{\psi \in \Gamma} \cup \ldots \cup Gen^{n}_{\psi \in \Gamma} \mid \text{no pair from } Gen^{n}_{\psi \in \Gamma} \text{ forms a family}\}.$$

In other words, $Gen^{n}_{\psi \in \Gamma}$ is the last generation if and only if no pair of members from $(\mathbf{AS})^{n}_{\psi \in \Gamma}$ constitutes a new family. We illustrate the concepts of *generation* and *offspring* in the example below.

EXAMPLE 5.7
Let $y : (p \to q), b : q, u : (o \to r)$ be formulas in Γ, and take $Gen^{1st}_{r \in \Gamma} = \{ASol_1, ASol_2, ASol_3\}$,

corresponding to the following *Sub*-minimal *ASol*s (with $a, b, u, \ldots, z \in V$).

$$\boxed{FirstGeneration:}$$

$$ASol_1.\ [x^{(q\to r)} * [y^{(p\to q)} * z^p]^q] : r$$

$$ASol_2.\ [a^{(q\to r)} * b^q] : r$$

$$ASol_3.\ [u^{(o\to r)} * [v^{(p\to o)} * w^p]^o] : r$$

Two families are formed by these *ASol*s. The first, $\mathcal{F}^{1,2}_{r\in\Gamma}$, is generated by the pair $(ASol_1, ASol_2)$; the second, $\mathcal{F}^{1,3}_{r\in\Gamma}$, is generated by the pair $(ASol_1, ASol_3)$. In turn, each of these families generates two new *Sub*-minimal *ASol*s, namely $ASol_4, ASol_5 \in \mathcal{F}^{1,2}_{r\in\Gamma}$ and $ASol_6, ASol_7 \in \mathcal{F}^{1,3}_{r\in\Gamma}$. Thus, $ASol_4$-$ASol_7$ constitute the second generation, as it is shown below.

$$\boxed{\mathcal{F}^{1,2}_{r\in\Gamma}.\ SecondGeneration:}$$

$$ASol_4.\ [x^{(q\to r)} * b^q] : r$$

$$ASol_5.\ [a^{(q\to r)} * [y^{(p\to q)} * z^p]^q] : r$$

$$\boxed{\mathcal{F}^{1,3}_{r\in\Gamma}.\ SecondGeneration:}$$

$$ASol_6.\ [x^{(q\to r)} * [y^{(p\to q)} * w^p]^q] : r$$

$$ASol_7.\ [u^{(o\to r)} * [v^{(p\to o)} * z^p]^o] : r$$

We can see that there is one more new *Sub*-minimal *ASol* that can be obtained from combining $ASol_5$ and $ASol_6$. In other words, the family $\mathcal{F}^{5,6}_{r\in\Gamma}$, with family heads $ASol_5$ and $ASol_6$, gives rise to a subsequent and final third generation:

$$\boxed{\mathcal{F}^{5,6}_{r\in\Gamma}.\ ThirdGeneration:}$$

$$ASol_8.\ [a^{(q\to r)} * [y^{(p\to q)} * w^p]^q] : r$$

Note: the only different *Sub*-minimal *ASol* available in $(Pre_{ASol_5,ASol_6,ASol_8})^{\mathsf{J}_{CS}}$ is their past relative, $ASol_1$ (see Theorem 5.5).

Finally, the offspring of *Sub*-minimal *ASol*s for the *AP* $r \in \Gamma$ is the union of all tree generations, i.e.

$$Off_{r\in\Gamma} = (\mathbf{AS})_{r\in\Gamma} \cup \{ASol_4, ASol_5, ASol_6, ASol_7\} \cup \{ASol_8\}$$

for $Gen^{1st}_{r\in\Gamma} = (\mathbf{AS})_{r\in\Gamma}$, $Gen^{2nd}_{r\in\Gamma} = \{ASol_4, ASol_5, ASol_6, ASol_7\}$ and $Gen^{3rd}_{r\in\Gamma} = \{ASol_8\}$.

5.2 Premise combination

In order to conclude our exposition regarding the phase of generation and classification in families, we give a more detailed account about how *Sub*-minimal *ASol* are bred. With this objective in mind, we present a series of examples, illustrating some of the possible types of combination of premises through which new *Sub*-minimal *ASol*s are created, and the qualities they have. Through all examples, let $y_n, z_n \in V$ for $n \in \mathbb{N}$.

- *Sharing*. Sharing is the basic method with which two *Sub*-minimal *ASol*s form a family; e.g. in Example 5.4, $[y_1 * [x_3 * x_4]] : o$ and $[x_1 * y_3] : o$ are obtained through the share of premises between $u : o$ and $s : o$. In fact some of the examples below are instances of sharing.
- *Reduction*. A *Sub*-minimal *ASol* generated by reduction is a shorter version of one of the heads constituting its family. Let $z_2 : a, z_3 : a \in \Gamma$. Also, let $D1$ and $D2$ be heads of family

$\mathcal{F}^{D1,D2}_{b\in\Gamma}$,

$$D1 = [z_1^{(a\to b)} * z_2^a] : b, \quad D2 = [[z_1^{(a\to(d\to b))} * z_3^a]^{(d\to b)} * z_4^d] : b$$

such that $D2$ contains a different interpretation of justification z_1 as a premise, $z_1 : (a \to (d \to b)) \in Pre_{D2}$. Then, we obtain a shorter version of $D2$:

$$D3 = [z_1^{(a\to b)} * z_3^a] : b \in \mathcal{F}^{D1,D2}_{b\in\Gamma}.$$

$D3$ is generated by the deletion of premises $[z_1 * z_3] : (d \to b), z_4 : d \in Pre_{D2}$, and the replacement of $z_1 : (a \to (d \to b)) \in Pre_{D2}$ with $z_1 : (a \to b) \in Pre_{D1}$. In other words, $D3$ is a reduced version of $D2$ in terms of *Op* value and, for the same reason, less explanative (i.e. it uses fewer premises in order to account for b).

– *Expansion*. Contrary to *reduction*, a *Sub*-minimal *ASol* produced by expansion is a longer a version of one of its heads. Following the example for reduction, $\mathcal{F}^{D1,D2}_{b\in\Gamma}$ also contains $D4$, which is a longer version of $D1$

$$D4 = [[z_1^{(a\to(d\to b))} * z_2^a]^{(d\to b)} * z_4^d] : b.$$

$D4$ is the opposite of $D3$; it is generated by extending $D1$ with premises from $D2$ and so, is better than $D1$ in terms of explanative power. In general, $D3$ and $D4$ are obtained through premise sharing between $D1$ and $D2$

– *Inversion*. With the aid of inversion a new *Sub*-minimal *ASol* is generated, in which the order of premises constituting one of the family heads is inverted. Let $\mathcal{F}^{I1,I2}_{b\in\Gamma}$ be a family with heads $I1$ and $I2$, such that $I1 = [[y_1 * y_2] * y_3] : n$, and $y_4 : (m \to (l \to n)) \in Pre_{I2}$. (Specifying the justification formulas belonging to Γ will not be relevant in this case).

$$I1 = [[y_1^{(l\to(m\to n))} * y_2^l]^{m\to n} * y_3^m] : n$$

Further, a substitution of $y_1 : (l \to (m \to n))$ with $y_4 : (m \to (l \to n))$ in Pre_{I1} leads to a new *Sub*-minimal *ASol* for n, $I3$.

$$I3 = [[y_4^{(m\to(l\to n))} * y_3^m]^{l\to n} * y_2^l] : n$$

$I3$ is a different accommodation of $y_2 : l, y_3 : m \in Pre_{I1}$, thus representing, e.g. an alternative chronological order in which $y_2 : l$ and $y_3 : m$ happened.

Naturally, new *Sub*-minimal *ASol*s can be generated by mixing two or more types of premise combination.

5.3 Selecting the most adaptable explanation

In addition to selecting a *Sub*-minimal *ASol* based on its comparison with every other available *Sub*-minimal *ASol* for the given *AP*, the framework of families allows for methods of evaluation of wider scope (when available *Sub*-minimal *ASol*s are able to form families). More specifically, as in the framework of families new *ASol*s are simultaneously generated and classified; this opens up the possibility of many methods of evaluation, with respect to a family, a generation or even a whole offspring.

Below we sketch a general method of evaluation, through which *Sub*-minimal *ASol*s can be selected, according to the relations they share with other members of an offspring, and the way they adapt to the present theory.

Example 5.8
As seen in Example 5.7, most of the offspring of *ASol*s for the *AP* $r \in \Gamma$, up until the only member of the third and last generation, is constituted by premises $a:(q \rightarrow r)$ or $x:(q \rightarrow r)$, which is an indication that these premises contain valuable information for the justification of r. In short, any *ASol* containing one of these premises could be highly considered as the best, or one of the best accounts for formula r. In other words, the most inherited properties and premises from generation to generation could be considered the most stable characteristics for an *ASol* to have. Consequently, *ASol*s with these characteristics may be more valuable than the rest.

On the other hand, *the best explanation* must also adapt to the theory. Given that theories tend to be ever-changing bodies of information, an *ASol* can be the best only to the extent that the theory remains unchanged. For example, if theory Γ changes in a way that it renders the information $(q \rightarrow r)$ as implausible (i.e. $(q \rightarrow r) \notin \Gamma$), most of the offspring of *Sub*-minimal *ASol*s for $r \in \Gamma$ will no longer adapt to the theory; thus, the majority of its members (i.e. the ones containing $a:(q \rightarrow r)$ or $x:(q \rightarrow r)$ as premise) will be wiped out. In this scenario, $ASol_3$ and $ASol_7$ are the only surviving solutions. Given Theorem 5.5, this pair cannot form a family and for this reason, new *Sub*-minimal *ASol*s, compatible with $ASol_3$ and/or $ASol_7$ must be found if one desires to obtain a new generation of *ASol*s for r.

6 Conclusions and further work

We have framed the meaning of abductive problem and abductive solutions in the language of justification logic. Our construction has the advantage that a solution is not only constituted by a set formulas from which to derive the unexplained formula, but also by a precise justification term for the formula. This term is a structurally explicit part of the solution and, thanks to the grounding of evidence by means of justification constants, it also exhibits clearly the evidence supporting it. Another advantage of our approach is the ability to go beyond of what can be done if we restrict ourselves to propositional logic, again thanks to the rich extra layer provided by justification terms.

Even if for each abductive problem there exist infinite solutions, our notion of minimality (*Sub*-miminality) allows to filter many of them, mostly the uninteresting ones. Also, in the final section of the paper we introduced a procedure designed to generate (‘breed’), organize and assess *Sub*-minimal solutions in groups of diverse scopes.

We believe that our formal apparatus can be put to use in a variety of abductive problems. Nonetheless, some refinements and extensions would make the proposal more practical, like delving deeper into the various methods of evaluation available in the framework of families. For example, we could extend the evaluation to whole groups of *Sub*-minimal solutions, be it families, generations or whole offsprings. Some of the possible criteria are *fruitfulness* (the number of members the group has), *plausibility* (how much of the information in the shared premises between members the group is true), *unification* (the number of formulas different to the abductive problem and relevant to the theory than can be justified with the group’s premises), among many others. We could even go one step further and evaluate the evolutive history of *Sub*-minimal abductive solutions, reconstructing the families from which their premises belong.

Finally, it should be mentioned that we are aware of different types of abduction [31, 38, 45] that could be translated into our evidential approach, and we would like to develop some of them in the future. For example, structural abduction, which is defined as the search for an adequate logic for achieving the derivation of some statement, shares a meaningful relation with justification logics, as Gabbay [14, 15] shows the usefulness of labels in formulas for codifying proofs of different logical nature in them. An interesting strategy of representation involves modal environments of structural abduction [36, 39], in which the notion of justification logics as '*the explicit counterparts of modal logics*' [3, pp. 497–499] could be capitalized in order to relate our work with these models.[20]

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References

[1] A. Aliseda. *Abductive Reasoning: Logical Investigations into Discovery and Explanations*. Springer, 2006.

[2] A. Aliseda. The logic of abduction: an introduction. In *Springer Handbook of Model-Based Science*, L. Magnani and T. Bertolotti, eds, pp. 219–230. Springer International Publishing, 2017. doi:10.1007/978-3-319-30526-4_10

[3] S. Artemov. The logic of justification. *The Review of Symbolic Logic*, **1**, 477–513, 2008. doi: 10.1017/S1755020308090060

[4] S. Artemov. Why do we need justification logic? In *Games, Norms and Reasons. Synthese Library*, J. van Benthem, A. Gupta and E. Pacuit, eds, pp 23–38. Studies in Epistemology, Logic, Methodology, and Philosophy of Science, vol. **353**. Springer, Dordrecht, 2011. doi:10.1007/978-94-007-0714-6_2

[5] S. Artemov and R. Kuznets. Logical Omniscience via Proof Complexity. In *Computer Science Logic, CSL 2006* Z. Ésik, (eds), pp. 135–149. Lecture Notes in Computer Science, vol **4207** Springer, Berlin, Heidelberg, 2006. doi:https://doi.org/10.1007/11874683_9

[6] C. Barés-Gómez and M. Fontaine. Argumentation and abduction in dialogical logic. In *Springer Handbook of Model-Based Science*, L. Magnani and T. Bertolotti, eds, pp. 295–314. Springer Handbooks. Springer, Cham, 2017. doi:10.1007/978-3-319-30526-4_14

[7] M. Beirlaen and A. Aliseda. A conditional logic for abduction. *Synthese*, **191**, 3733–3758, 2014. doi:10.1007/s11229-014-0496-0

[8] J. Bueno-Soler, W. Carnielli, M. E. Coniglio and F. A. Rodrigues. Formal (in)consistency, abduction and modalities. In *Springer Handbook of Model-Based Science* , L. Magnani and T. Bertolotti, eds, pp. 315–335. Springer Handbooks. Springer, Cham, 2017. doi:10.1007/978-3-319-30526-4_15

[9] P. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and *n*-person games. *Artificial Intelligence*, **77**, 321–357, 1995. doi:10.1016/0004-3702(94)00041-X

[10] L. Estrada-González. Remarks on some general features of abduction. *Journal of Logic and Computation*, **23**, 181–197, 2013. doi:https://doi.org/10.1093/logcom/exs005

[11] M. Fitting. Justification logics and realization. *Computer Science Technical Reports*. Paper 395, CUNY Academic Works, 2014. https://academicworks.cuny.edu/gc_cs_tr/395

[12] P. A. Flach. On the logic of hypothesis generation. In *Abduction and Induction. Essays on their*

[20] A more straightforward way of representation lies in the *CS* set: a justification logic J_{CS} can be expanded with new axioms in order to form a new justification logic $\mathsf{J}_{CS}{}^{*}$, under which the statement in question (our *AP*) could be derived.

Relation and Integration, P. A. Flach and A. C. Kakas, eds, pp. 89–106. Springer, Netherlands, 2000. doi:10.1007/978-94-017-0606-3_6

[13] P. A. Flach and A. C. Kakas, eds. *Abduction and Induction. Essays on their Relation and Integration*. Springer, Netherlands, 2000. doi:10.1007/978-94-017-0606-3

[14] D. M. Gabbay. *Labelled Deductive Systems*. Oxford University Press, 1996.

[15] D. M. Gabbay. Abduction in labelled deductive systems. In *Abductive Reasoning and Learning*, D. M. Gabbay and R. Kruse, eds, pp. 99–154. Handbook of Defeasible Reasoning and Uncertainty Management Systems, vol. **4**. Springer, Dordrecht, 2000. doi:10.1007/978-94-017-1733-5_3

[16] D. M. Gabbay and J. Woods. *A Practical Logic of Cognitive Systems* Volume 2. The Reach of Abduction: Insight and Trial. Elsevier, 2005.

[17] T. Gauderis. Modelling abduction in science by means of a modal adaptive logic. *Foundations of Science*, **18**, 611–624, 2013. doi:10.1007/s10699-012-9293-8

[18] D. Grossi. On the logic of argumentation theory. In *AAMAS'10 Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems*, W. van der Hoek, G. A. Kaminka, Y. Lespérance, M. Luck, S. Sen, eds, vol. 1, pp. 409–416. IFAAMAS, 2010.

[19] D. Grossi. Argumentation in the view of modal logic. In *Argumentation in Multi-Agent Systems. ArgMAS 2010* , P. McBurney, I. Rahwan and S. Parsons, eds, pp. 190–208. Lecture Notes in Computer Science, vol. **6614**. Springer, Berlin, Heidelberg, 2011. doi:10.1007/978-3-642-21940-5_12

[20] G. H. Harman. The inference to the best explanation. *The Philosophical Review*, **74**, 88–95, 1965. https://www.jstor.org/stable/2183532?origin=crossref&seq=1#page_scan_tab_contents

[21] J. Hintikka. What is abduction? The fundamental problem of contemporary epistemology. *Transactions of the Charles S. Peirce Society*, **34**, 503–533, 1998. https://www.jstor.org/stable/40320712?seq=1#page_scan_tab_contents

[22] J. Hintikka. Interrogative logic as a general theory of reasoning. *Inquiry as Inquiry: A Logic of Scientific Discovery*, pp. 47–90. Jaakko Hintikka Selected Papers, vol. **5**. Springer, Dordrecht, 1999. doi:10.1007/978-94-015-9313-7_3

[23] J. Hintikka. Abduction—inference, conjecture, or an answer to a question? *Socratic Epistemology. Explorations of Knowledge-Seeking by Questioning*, pp. 38–60. Cambridge University Press, 2007. doi:10.1017/CBO9780511619298

[24] V. Iranzo. Abduction and inference to the best explanation. *THEORIA, Revista de Teoría, Historia Y Fundamentos de la Ciencia*, **22**, 339–346, 2007. doi:10.1387/theoria.455

[25] J. R. Josephson. Smart inductive generalizations are abductions. In *Abduction and Induction. Essays on their Relation and Integration*, P. A. Flach and A. C. Kakas, eds, pp. 31–44. Springer, Netherlands, 2000. doi:10.1007/978-94-017-0606-3_2

[26] T. A. F. Kuipers. Abduction aiming at empirical progress or even-truth approximation leading to a challenge for computational modelling. *Foundations of Science*, **4**, 307–323, 1999. doi:10.1023/A:1009688227714

[27] R. Kuznets and T. Studer. Justifications, ontology, and conservativity. In *Advances in Modal Logic* , T. Bolander, T. Brauner, S. Ghilardi and L. S. Moss, eds, pp. 437–458. College Publications, 2012.

[28] P. Lipton. *Inference to the Best Explanation*, 2nd edn. Routledge, London, 2004.

[29] M. Ma and A. V. Pietarinen. A dynamic approach to Peirce's interrogative construal of abductive logic. In *The IfCoLog Journal of Logics and their Applications. Frontiers of Abduction*, L. Magnani ed, pp. 73–104. College Publications, 2016.

[30] L. Magnani. *Abduction, Reason and Science. Processes of Discovery and Explanation*. Springer, US, 2001. doi:10.1007/978-1-4419-8562-0

[31] L . Magnani. *Abductive Cognition. The Epistemological and Eco-Cognitive Dimensions of Hypothetical Reasoning*. Springer, Berlin, Heidelberg, 2009. doi:10.1007/978-3-642-03631-6

[32] L. Magnani. The eco-cognitive model of abduction. 'Apagoge' now: naturalizing the logic of abduction. *Journal of Applied Logic*, **13**, 285–315, 2015. (In Greek) doi:10.1016/j.jal.2015.04.003

[33] L. Magnani. The eco-cognitive model of abduction II: irrelevance and implausibility exculpated. *Journal of Applied Logic*, **15**, 94–129, 2016. doi:10.1016/j.jal.2016.02.001

[34] J. Meheus and D. Batens. A formal logic for abductive reasoning. *Logic Journal of the IGPL*, **14**, 221–236, 2006. doi:10.1093/jigpal/jzk015

[35] J. Meheus and D. Provijn. Abduction through semantic tableaux versus abduction through goal-directed proofs. THEORIA. *Revista de Teoría, Historia y Fundamentos de la Ciencia*, **22**, 295–304, 2007. doi:10.1387/theoria.450

[36] A. Nepomuceno-Fernández, F. J. Salguero-Lamillar and D. Fernández-Duque. Tableaux for structural abduction. *Logic Journal of the IGPL*, **20**, 388–399, 2012. doi:10.1093/jigpal/jzq054

[37] I. Niiniluoto. Peirce, abduction and scientific realism. In *Ideas in Action: Proceedings of the Applying Peirce Conference* , M. Bergman, S. Paavola, A. Pietarinen and H. Rydenfelt, eds, pp. 252–263. Nordic Pragamatism Network, 2010.

[38] G. Schurz. Patterns of abduction. *Synthese*, **164**, 201–234, 2008. doi:10.1007/s11229-007-9223-4

[39] F. Soler-Toscano, D. Fernández-Duque and A. Nepomuceno-Fernández. A modal framework for modelling abductive reasoning. *Logic Journal of the IGPL*, **20**, 438–444, 2012. doi:10.1093/jigpal/jzq059

[40] F. Soler-Toscano and F. R. Velázquez-Quesada. Generation and selection of abductive explanations for non-omniscient agents. *Journal of Logic, Language and Information*, **23**, 141–168, 2014. doi:10.1007/s10849-014-9192-1

[41] T. Studer. Lectures on justification logic. http://home.inf.unibe.ch tstuder/papers/JL_Book.pdf

[42] P. R. Thagard. The best explanation. Criteria for theory choice. *The Journal of Philosophy*, **75**, 76–92, 1978. doi:10.2307/2025686

[43] F. Tohmé, G. Caterina and R. Gangle. Abduction: a categorical characterization. *Journal of Applied Logic*, **13**, 78–90, 2015. doi:10.1016/j.jal.2014.12.004

[44] F. R. Velázquez-Quesada, F. Soler-Toscano and A. Nepomuceno-Fernández. An epistemic and dynamic approach to abductive reasoning: abductive problem and abductive solution. *Journal of Applied Logic*, **11**, 505–522, 2013. doi:10.1016/j.jal.2013.07.002

[45] P. Woosuk. On classifying abduction. *Abduction in Context. The Conjectural Dynamics of Scientific Reasoning*, pp. 9–39. Springer, 2017. doi:10.1007/978-3-319-48956-8_2

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