ADC method of proof search for intuitionistic propositional natural deduction

G. MINTS and SH. STEINERT-THRELKELD, *Department of Philosophy, Stanford University, Stanford, CA 94305.*
E-mail: gmints@stanford.edu; shanest@stanford.edu

Abstract

The ADC method of proof search in propositional natural deduction proposed in 2000 proceeds bottom up first by *Analysing* the sequent into sub-goals by applying all possible introduction rules, and then by checking whether each of these sub-goals can be established using only elimination rules (*Direct Chaining*). This looks simpler than the worst case complexity (PSPACE) of the derivability problem for intuitionistic propositional logic. We investigate the complexity of ADC for various fragments. ADC derivability is polynomially decidable for the &, →-fragment by a generalization of a familiar direct chaining algorithm for Horn formulas. Adding ∨ leads to a CoNP-complete fragment. A short counterexample in case the goal sequent is not ADC derivable is provided by witnessing all relevant disjunctions. Adding constant ⊥ preserves CoNP completeness.

Keywords: Natural deduction, intuitionistic logic, polynomial decidability.

1 Introduction

In this article, we investigate a method of proof search originally proposed in [4] as a pedagogical device to make it easier for students to construct natural deductions. With a renewed interest in polynomially decidable classes of formulas—especially in applied problems (see e.g. [2], a mild correction of [3])—this method acquires additional interest.

Given a goal sequent, this method proceeds bottom up first by *analysing* the sequent into sub-goals by applying all possible introduction rules, and then by checking whether each of these sub-goals can be established using only elimination rules. This latter step will be called *direct chaining*. So we refer to the whole method as ADC for analysis and direct chaining.

1.1 System NJp

Natural deduction is formulated here in terms of *sequents*

$$A_1,\ldots,A_n \Rightarrow B \tag{1}$$

where $A_1,\ldots,A_n,B$ are propositional formulas constructed from propositional variables $p,q,r,p_1,\ldots$ and the constant $\bot$ (false) by $\&,\to,\vee$. Negation $\neg A$ can be defined as $A\to\bot$. A sequent (1) is read: 'B follows from assumptions $A_1,\ldots,A_n$'.

The *antecedent* $A_1,\ldots,A_n$ of a sequent (1) is treated as a set of formulas, so contraction of identical formulas and permutation in the antecedent are implicitly included in the rules below. Finite sets of formulas are denoted Γ,Δ,Σ, etc. Axioms and inference rules of *intuitionistic propositional natural deduction system* NJp are presented below. $\mathbf{c}^+,\mathbf{c}^-$ indicate introduction and elimination rule for a connective $\mathbf{c}$.

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$$\text{Axioms} \qquad A,\Gamma \Rightarrow A$$

$$\frac{A,\Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B} \to^+ \qquad \frac{\Gamma \Rightarrow A \to B \quad \Delta \Rightarrow A}{\Gamma,\Delta \Rightarrow B} \to^-$$

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma,\Delta \Rightarrow A\&B} \&^+ \qquad \frac{\Gamma \Rightarrow A\&B}{\Gamma \Rightarrow A} \&^- \frac{\Gamma \Rightarrow A\&B}{\Gamma \Rightarrow B}$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \vee^+ \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$\frac{\Gamma \Rightarrow A \vee B \quad A,\Delta \Rightarrow C \quad B,\Sigma \Rightarrow C}{\Gamma,\Delta,\Sigma \Rightarrow C} \vee^- \qquad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A} \bot,$$

where A is a propositional formula.

As is well-known, our formulation of the axiom makes the weakening rule

$$\frac{\Gamma \Rightarrow A}{C,\Gamma \Rightarrow A}$$

admissible.

Note that the ADC method is incomplete. For example, the sequent

$$p \vee q \Rightarrow q \vee p$$

obviously cannot be derived so that $\vee^+$ is the last rule in the derivation.

Another example is the sequent $\Rightarrow (p\&q \to r) \to (p \to (q \to r))$. This will first be analysed to $p\&q \to r, p, q \Rightarrow r$, which is derivable as follows:

$$\dfrac{\dfrac{p \Rightarrow p \qquad q \Rightarrow q}{p,q \Rightarrow p\&q} \qquad p\&q \to r \Rightarrow p\&q \to r}{p\&q \to r, p, q \Rightarrow r}$$

This sequent, however, will not be derivable using ADC, as the conjunction in the antecedent of the conditional $p\&q \to r$ must be introduced before the $\to$ is eliminated. However, direct chaining does not allow introduction rules to be applied.

As an example of the ADC search procedure, consider the sequent (and recall that $\neg q := q \to \bot$)

$$\Rightarrow p\&q \to \neg(p \to \neg q)$$

Two steps of analysis generates the new goal sequent

$$p\&q, p \to \neg q \Rightarrow \bot$$

To see whether this sequent is derivable by direct chaining, we *saturate* $\Gamma = \{p\&q, p \to \neg q\}$ with the conclusions of all applicable elimination rules (i.e. elimination rules where the consequents of the premises are in Γ). This saturation procedure generates , in order, the following sets:

$$\Gamma_1 = \{p\&q, p \to \neg q, p, q\}$$
$$\Gamma_2 = \{p\&q, p \to \neg q, p, q, q \to \bot\}$$
$$\Gamma_3 = \{p\&q, p \to \neg q, p, q, q \to \bot, \bot\}$$

Because $\bot \in \Gamma_3$, we know that the goal sequent generated by the analysis procedure is derivable by direct chaining and so our original sequent is ADC derivable.

In what follows, we investigate the complexity of ADC for various fragments of propositional intuitionistic natural deduction. It turns out that ADC derivability is polynomially decidable for the $\&, \rightarrow$-fragment by a generalization of a familiar direct chaining algorithm for Horn formulas.

Specific features of intuitionistic logic are important here, as already $\rightarrow$-fragment contains Pierce law $((p \rightarrow q) \rightarrow p) \rightarrow p$, which is derivable classically but not intuitionistically. Proof search for classical natural deduction attracted attention for a long time [5, 6] and most recently [7].

Adding $\vee$ leads to CoNP-complete fragment. A short counterexample in case the goal sequent is not derivable is provided by witnessing all relevant disjunctions. Adding constant $\bot$ preserves CoNP-completeness.

After preliminary definitions and precise description of the proof search method in section 2, we consider $\&, \rightarrow$-fragment in section 3 , $\&, \rightarrow, \vee$-fragment in section 4 and complete propositional language in section 5.

Section 6 restates some of the previous material in terms of Gentzen-style rules. This simplifies some of the proofs and constructions or at least throws some new light. We then briefly compare in section 7 our search procedure with that of Sieg, which is complete and therefore of greater complexity.

2 Preliminaries

DEFINITION 2.1
ADC is the set of natural deductions where no introduction inference precedes an elimination inference.

Note that all ADC deductions are normal in the sense that no introduction rule is followed by an elimination rule. Some care will have to be taken to ensure normality when we add $\bot$.

DEFINITION 2.2
A *DC* derivation is a derivation in NJp by elimination rules.

$\vdash_{DC} \Gamma \Rightarrow A$ means that the sequent $\Gamma \Rightarrow A$ has a *DC* derivation. Similarly for $\vdash_{ADC}$. Notation like $\vdash_{DC\&,\rightarrow}$ means that only rules $\&^-, \rightarrow^-$ are allowed.

An ADC derivation written as a tree of sequents consists of two parts. The bottom part (analysis) ending in the last sequent of the derivation is a deduction by introduction rules only. The maximal height of a branch in that part is bounded by the number of connectives in the goal formula. Therefore, the main contribution to complexity comes from the upper DC part, which will be the main object of our attention.

If $ADC \vdash S$ then S is a tautology, as all rules of natural deduction are sound.

2.1 Analysis part of ADC

This part unfolds the goal formula G in the sequent

$$\Gamma \Rightarrow G \tag{2}$$

applying introduction rules bottom up to G. The process and the result are more complicated when the goal formula G contains positive occurrences of disjunction (we say 'contains $+\vee$' for short). In all cases, the length of the process is bounded by the number of connectives in the goal formula G.

2.1.1 G does not contain $+\vee$

In this case, the result of the process is a tree T of sequents, the *analysis tree* of (2). *Final* leaves of T are by definition leaves containing non-axiom sequents with atomic formulas in the succedent. Each of them will be an initial datum for the DC part of the ADC method. The ADC process is successful if *all* final leaves of T are DC derivable.

The analysis tree is defined inductively via a sequence of trees T_i. T_0 is the one-point tree containing only the sequent (2), which is the only leaf of that tree. If all leaves of T_i are final, then the tree itself is *final* and the analysis part is finished. If all leaves of T are axioms, then T is *closed*. In this case, the whole ADC process ends successfully: the goal (2) is derivable by introduction rules only.

Otherwise take the leftmost non-final leaf of T_i

$$\Sigma \Rightarrow A, \tag{3}$$

then analyse it bottom up by the introduction rule for A.

If $A = B\&C$ then T_{i+1} is obtained by writing over (3) two new leaves:

$$\begin{array}{ccc} \Sigma \Rightarrow B & & \Sigma \Rightarrow C \\ & \nwarrow \ \nearrow & \\ & \Sigma \Rightarrow B\&C & \end{array}$$

Note that although the $\&^{+}$ rule for NJp is formulated multiplicatively in that each premise is treated as having a distinct context, the admissibility of weakening makes this analysis acceptable.

If $A = B \rightarrow C$ then one new leaf is added:

$$\begin{array}{c} \Sigma, B \rightarrow C \\ \uparrow \\ \Sigma \Rightarrow B \rightarrow C \end{array}$$

2.1.2 G contains $+\vee$

In this case, the final result of analysis is in general a finite set

$$T_{f1} \ldots, T_{fn} \tag{4}$$

of final trees treated as *parallel goals* in subsequent DC part. The ADC process is successful if all final leaves of *one* of the trees T_{fi} are DC derivable.

More precisely, at each stage $s = 1, 2, \ldots$ of the analysis part, we have a *forest*

$$T_{s1}, \ldots, T_{sn_s} \tag{5}$$

consisting of a finite number of trees. For $s = 1$, we have as before just one tree consisting of the goal sequent (2).

If each of $T_{sj}, 1 \leq j \leq n_s$ is a final tree, the analysis stage is finished. If *at least one* of T_{sj} is closed then the whole ADC process ends successfully: the goal (2) is derivable by introduction rules only.

Otherwise take the first non-closed non-final tree among T_{sj}, say T_{sk} and a non-final leaf sequent (3) in it. The resulting forest at the stage $s+1$ is obtained by applying a tree extension step to this leaf.

If the main connective of A is & or $\to$, then apply the same tree extension step as before to T_{sk} and replace T_{sk} by the result T'_{sk} in (5) to get

$$T_{(s+1)i} = \begin{cases} T_{s,i} & i \neq k \\ T'_{sk} & i = k \end{cases}$$

If $A = (B_1 \vee B_2)$ then apply the tree extension steps for B_1 and for B_2 (in parallel) to T_{sk}. In other words, form two trees $T^j_{sk}, j = 1,2$ by replacing the leaf (3) in T_{sk} by $\Sigma \Rightarrow B_j$. Then replace T_{sk} in (5) by two trees T^1_{sk} and T^2_{sk} resulting in

$$T_{(s+1)i} = \begin{cases} T_{s,i} & i \neq k, i \leq n_s \\ T^1_{sk} & i = k \\ T^2_{sk} & i = n_s + 1 \end{cases}$$

This concludes the description of the Analysis part of ADC.

Let's estimate the complexity of this part. The length of each sequent in each of the trees T_{si} is bounded by the length l of the goal sequent. The number of leaves in each of these trees is also bounded by l, or more precisely by the number c of connectives in the goal formula. The number n_s of these trees in (5) is also bounded by c, or more precisely by the number d of positive disjunctions in the goal formula. At each stage s it is enough to store only leaves of the trees T_{si}. Hence, the total amount of information to be passed from the Analysis stage to the DC stage is bounded by $l \times c \times d \leq l^3$.

2.2 *DC part of ADC*

Let's describe the DC part of the ADC method. The presence of disjunction again creates complications, but in all cases, the length of the process is bounded, as each non-trivial step only adds some subformulas of the goal sequent.

DEFINITION 2.3

(&, $\to$)-closure Γ_+ of a finite set Γ of formulas is defined as a closure of Γ under the rules $\&^-, \to^-$. More precisely, one-step &, $\to$-closure is defined as

$$\Gamma^\circ := \Gamma \cup \{C\colon (A\&C) \in \Gamma \text{ or } (C\&A) \in \Gamma \text{ or } \{A, A \to C\} \subseteq \Gamma\}$$

for some A. After this

$$\Gamma_+ := \bigcup_n \Gamma^n$$

where $\Gamma^0 = \Gamma$ and $\Gamma^{n+1} = (\Gamma^n)^\circ$.

The (&, $\to$, $\vee$)-closure $\Gamma_\vee$ is defined similarly, but at each stage the rule $\vee^-$ is applied bottom up to each $\vee$-formula obtained at previous stage:

$$\begin{array}{c} C, \Delta \Rightarrow G \quad D, \Delta \Rightarrow G \\ \nwarrow\!\nearrow \\ \Delta \Rightarrow G \end{array} \qquad\qquad (C \vee D) \in \Delta.$$

DEFINITION 2.4

$$\Gamma_{\vee} := \bigcup_{n} \Gamma_{\vee}^{n}, \text{ where } \qquad \Gamma_{\vee}^{0} := \Gamma$$

$$\Gamma_{\vee}^{n+1} := (\Gamma_{\vee}^{n}) \cup \bigcup \{(C,(\Gamma^{n})_{\vee})_{+} \cap (D,(\Gamma^{n})_{\vee})_{+} : (C \vee D) \in (\Gamma^{n})_{\vee})\}$$

DEFINITION 2.5
The $(\bot)$-closure $\Gamma_{\bot}$ is the set of all subformulas of Γ, if $\bot \in \Gamma$, and is Γ itself if $\bot \notin \Gamma$.

DEFINITION 2.6
Finally the $(\&, \rightarrow, \vee, \bot)$-closure $\Gamma_{\vee,\bot}$ is defined by

$$\Gamma_{\vee,\bot} := \bigcup_{n} \Gamma_{\vee,\bot}^{n}, \text{ where } \qquad \Gamma_{\vee,\bot}^{0} := \Gamma$$

$$\Gamma_{\vee,\bot}^{n+1} := \Gamma_{\vee,\bot}^{n} \cup \bigcup \{((C,\Gamma^{n})_{\vee,\bot})_{\bot} \cap ((D,\Gamma^{n})_{\vee,\bot})_{\bot} : \ (A \vee B) \in \Gamma_{\vee,\bot}^{n})\}$$

DEFINITION 2.7
The DC method for a goal sequent $\Gamma \Rightarrow A$ is *successful* if $A \in \Gamma_{\vee,\bot}$ or $\bot \in \Gamma_{\vee,\bot}$.

ADC method for a goal sequent $\Gamma \Rightarrow A$ is *successful* if DC method is successful for all final leaves of at least one of the trees in the forest, which is the final result of the Analysis process applied to $\Gamma \Rightarrow A$.

2.3 Soundness and completeness

LEMMA 2.8 (soundness)

1. If a sequent is derivable by introduction rules then Analysis part for this sequent is successful.
2. If a sequent is DC derivable then DC method with this goal sequent is successful.
3. If a sequent is ADC derivable then the ADC method with this goal sequent is successful.

PROOF. The proof for all three cases is by induction on the height N of the derivation. If $N = 1$ then the goal sequent is an axiom, and the search process terminates immediately. Otherwise apply induction hypothesis. If for example the last rule in a given derivation was $\vee^{-}$

$$\frac{\Gamma \Rightarrow C \vee D \quad C,\Gamma \Rightarrow G \quad D,\Gamma \Rightarrow G}{\Gamma \Rightarrow G}$$

then by induction hypothesis all premises are in $\Gamma_{\vee,\bot}^{n}$ for some n, therefore the conclusion is in $\Gamma_{\vee,\bot}^{n+1}$. ■

For $(\bot)$-inference note that if $\Gamma \Rightarrow C$ is DC derivable, then C is a subformula of Γ (Lemma 5.4 below).

THEOREM 2.9
If ADC method for a sequent is successful then this sequent is ADC-derivable.

PROOF. It is assumed that at least one of the final trees T_{fj} in (4) is successful, i.e. DC part is successful for each final leaf of this tree. Induction on the number of steps in the DC part shows that each of these leaves has a DC derivation. Then similar induction on the height of the tree T_{fj} shows that the goal sequent has an ADC derivation. ■

3 $\rightarrow$, &-fragment

Consider first $\rightarrow$, &-fragment of NJp.

DEFINITION 3.1
An occurrence of a subformula in a formula A is *strictly positive* if it is not in the antecedent of any implication.

THEOREM 3.2
The relation $\Gamma \vdash_{DC\&,\rightarrow} A$ is polynomially decidable.

PROOF. First, it is easy to prove by induction on a derivation d establishing $\Gamma \vdash_{DC\&,\rightarrow} A$ that any sequent in d has a form

$$\Gamma' \Rightarrow A' \tag{6}$$

where $\Gamma' \subseteq \Gamma$ and A' is a strictly positive subformula of (some formula in) Γ.

Now consider an arbitrary DC derivation d of $\Gamma \Rightarrow A$ *in linear form*, i.e. a sequence of formulas such that each of them is a member of Γ or is obtained from previous ones by an inference rule. Assuming that d does not contain repetitions of formulas, the length of d is bounded by $g+p$, where g is the number of formulas in Γ and p is the number of strictly positive subformulas of Γ.

So the process of Direct Chaining (adding results of all possible inferences by elimination rules) stops after p steps. ■

NOTE. It is not true that every DC-provable formula is a substitution instance of a derivable Horn formula. A counterexample:

$$(((a \rightarrow b) \rightarrow c)\&(a \rightarrow b)\&a\&(b \rightarrow (c \rightarrow d))) \rightarrow d$$

Therefore, providing a syntactic characterization of the (A)DC-derivable formulas remains an interesting open problem.

4 Adding disjunction

Consider now the $(\&, \rightarrow, \vee)$ fragment of NJp, that is the $\bot$-free fragment.

Let's state first a refinement of a well-known subformula property [4, p. 41].

LEMMA 4.1
Every $DC_{\&,\rightarrow,\vee}$-derivation of a sequent

$$\Gamma \Rightarrow A$$

consists of sequents

$$\Delta, \Gamma' \Rightarrow P, \tag{7}$$

where $\Gamma' \subseteq \Gamma$, P occurs strictly positively in Γ and for every $\delta \in \Delta$ there is a formula $C \vee D$ occurring strictly positively in Γ and such that $\delta = (C \vee D)$ or $\delta = C$ or $\delta = D$.

In particular A occurs strictly positively in Γ.

PROOF. Induction on the derivation. Consider the principal branch, i.e. the branch containing the endsequent and the left hand side (principal) premise of any rule. The principal branch ends in the endsequent of the derivation and begins in an axiom $\gamma \Rightarrow \gamma$. Tracing the principal branch from the bottom up, we see that every sequent in that branch has a form (7) with empty Δ (hence $\gamma \in \Gamma$) and

P strictly positive in Γ. In particular, principal formulas of all $\vee^-$-inferences in the principal branch are strictly positive subformulas of Γ. Now apply induction hypothesis to side premises of the rules in the principal branch. ∎

Next, we embed an NP-complete class of propositional formulas into our fragment. Recall that our current fragment does not contain negation, but enough of negation can be defined 'locally'.

Let a propositional variable p be fixed and denote

$$\neg_p a := (a \rightarrow p)$$

DEFINITION 4.2
A *clause* is a disjunction of literals, i.e. formulas of the form $a, \neg_p a$, where a is an atomic formula.

LEMMA 4.3
$C(p) \rightarrow p$ is a tautology iff $C(\bot)$ is inconsistent, for every formula $C(p)$.

PROOF. If $C(p) \rightarrow p$ is a tautology then substitute $\bot$ for p.

If $C(\bot) \rightarrow \bot$ is a tautology, then $C(p) \rightarrow p$ is a tautology, as $C(\top) \rightarrow \top$ (always) is. ∎

LEMMA 4.4
Let Γ be a finite set of clauses. Then a sequent $\Gamma \Rightarrow p$ has a DC-derivation iff it is a tautology. Therefore, DC derivability is CoNP hard.

PROOF. If a sequent is derivable, then it is a tautology, as all inference rules are sound. Let now $\Gamma \Rightarrow p$ be a tautology. We construct a DC derivation by induction on the number n of disjunctions in Γ.

Induction base $n = 0$. Then Γ consists of clauses of the form $a, \neg_p b$. Therefore, our sequent has one of two forms

$$\Gamma', p \Rightarrow p, \qquad \Gamma', a, a \rightarrow p \Rightarrow p.$$

Indeed otherwise it is not a tautology: assign the truth-value $\bot$ to p and all variables b such that Γ contains $\neg_p b$. Assign the value $\top$ to all other variables.

Induction step. $\Gamma \equiv (A \vee B), \Gamma'$. Apply induction hypothesis to get DC derivations of $A, \Gamma' \Rightarrow p$ and $B, \Gamma' \Rightarrow p$, as both sequents are tautologies. Then apply $\vee^-$:

$$\frac{A \vee B \Rightarrow A \vee B \quad A, \Gamma' \Rightarrow p \quad B, \Gamma' \Rightarrow p}{A \vee B, \Gamma' \Rightarrow p}$$

∎

THEOREM 4.5
The problem of *DC* derivability of arbitrary propositional $\bot$-free sequents is CoNP-complete.

PROOF. One direction is the previous Lemma. For the other direction, we give in the next two lemmata, a polynomially checkable condition of non-derivability, showing that DC derivability is in CoNP. ∎

DEFINITION 4.6
A sequence $\mathcal{R}$ of formulas is called a *DC* refutation of a sequent $\Gamma \Rightarrow A$ if

1. $\mathcal{R}$ consists of strictly positive subformulas of Γ,
2. $\mathcal{R} \supseteq \Gamma$ but $A \notin \mathcal{R}$,

3. $\mathcal{R}$ is $DC_{\&\rightarrow}$-closed,
4. if $(C \vee D) \in \mathcal{R}$ then $C \in \mathcal{R}$ or $D \in \mathcal{R}$.

A similar notion was used previously in [1]. Note that the size of a minimal DC refutation is bounded by the number of strictly positive subformulas of Γ.

LEMMA 4.7
If a sequent $\Gamma \Rightarrow A$ is not DC derivable then there is a DC refutation of $\Gamma \Rightarrow A$.

PROOF. Induction on the number $n(\Gamma)$ of strictly positive subformulas $C \vee D$ of Γ undecided by Γ, i.e. such that neither of C, D is a member of Γ. Induction base is trivial.

For induction step take the $DC_{\&\rightarrow}$-closure Γ_+ of Γ.

We still have $\Gamma_+ \nvdash_{DC} A$. Note that $n(\Gamma_+) \leq n(\Gamma)$, as only strictly positive subformulas were added. Take strictly positive subformula

$$(C_1 \vee C_2) \in \Gamma_+$$

undecided by Γ_+. (If there is no such formula then Γ_+ is already a DC refutation). One of the sequents $C_i, \Gamma_+ \Rightarrow A$, $i \in \{1,2\}$ is DC underivable, as otherwise $\Gamma_+ \Rightarrow A$ is obtained by $\vee$-elimination. Then apply the induction hypothesis to $C_i, \Gamma_+ \Rightarrow A$. ■

LEMMA 4.8
(a) Every DC refutation $\mathcal{R}$ is DC deductively closed:

$$\mathcal{R} \vdash_{DC\&\rightarrow\vee} G \text{ implies } G \in \mathcal{R}$$

(b) If there is a DC refutation of $\Gamma \Rightarrow A$ then $\Gamma \nvdash_{DC\&\rightarrow\vee} A$.

PROOF. (b) immediately follows from (a). (a) is proved by induction on a DC derivation. Induction base is obvious. For induction step consider the last inference. For rules other than $\vee^-$ apply induction hypothesis and closure of $\mathcal{R}$ under these rules. For $\vee^-$

$$\frac{\mathcal{R} \Rightarrow C_1 \vee C_2 \quad C_1, \mathcal{R}, \Gamma \Rightarrow G \quad C_2, \mathcal{R}, \Gamma \Rightarrow G}{\mathcal{R} \Rightarrow G}$$

apply the induction hypothesis to $\mathcal{R} \Rightarrow C_1 \vee C_2$ to get $(C_1 \vee C_2) \in \mathcal{R}$, then $C_i \in \mathcal{R}$ for some $i \in \{1,2\}$. This means $C_i, \mathcal{R} = \mathcal{R}$; therefore, induction hypothesis is applicable to the premise $C_i, \mathcal{R} \Rightarrow G$. ■

5 Adding negation

Now, we add the last remaining connective $\perp$ to the language.

DEFINITION 5.1
A DC deduction is redefined now as a natural deduction by $(\perp)$ and remaining elimination rules

$$\&^-, \rightarrow^-, \vee^-. \tag{8}$$

Recall that such a deduction is *normal* if no conclusion of $(\perp)$ is a major premise of any elimination rule, including another $(\perp)$.

DEFINITION 5.2
Call a DC deduction *pruned* if it is normal and all $(\bot)$-inferences are pushed maximally down. In other words, we add to conversions used for normalization further conversions:

$$\dfrac{\Pi \Rightarrow A \to B \quad \dfrac{\Phi \Rightarrow \bot}{\Phi \Rightarrow A}}{\Pi, \Phi \Rightarrow B} \text{ conv } \quad \dfrac{\Phi \Rightarrow \bot}{\Pi, \Phi \Rightarrow B}$$

$$\dfrac{\Pi \Rightarrow A \vee B \quad \dfrac{A, \Phi \Rightarrow \bot}{A, \Phi \Rightarrow C} \quad \dfrac{B, \Psi \Rightarrow \bot}{B, \Psi \Rightarrow C}}{\Pi, \Phi, \Psi \Rightarrow C} \text{ conv } \quad \dfrac{\dfrac{\Pi \Rightarrow A \vee B \quad A, \Phi \Rightarrow \bot \quad B, \Psi \Rightarrow \bot}{\Pi, \Phi, \Psi \Rightarrow \bot}}{\Pi, \Phi, \Psi \Rightarrow C}$$

LEMMA 5.3
Any DC deduction can be easily normalized and pruned by pushing down and removing some $\bot$-inferences. Resulting deduction is shorter than the original one (unless it is unchanged).

We modify Lemma 4.1 for DC deductions, then modify Definition 4.6 so that the proofs of Lemmata 4.7, 4.8 still go through.

LEMMA 5.4
For every sequent

$$\Sigma \Rightarrow F \tag{9}$$

in a pruned *DC* deduction of a sequent $\Gamma \Rightarrow A$ we have

$$\Sigma = \Delta, \Gamma' \tag{10}$$

where $\Gamma' \subseteq \Gamma$ and for every $\delta \in \Delta$ there is a formula $C \vee D$ occurring strictly positively in Γ with $\delta = (C \vee D)$ or $\delta = C$ or $\delta = D$, and one of the following three conditions holds.

(1) (9) is not a conclusion of $(\bot)$ and F is strictly positive in Γ,
(2) (9) is a conclusion of $(\bot)$ and F is negative in Γ or
(3) (9) is a conclusion of $(\bot)$ introducing the endsequent, and hence $F = A$.

PROOF. Induction on the derivation. Consider the principal branch, i.e. the branch containing the endsequent and the left hand side (principal) premise of any rule. If this branch contains a $(\bot)$-inference, then this inference introduces the endsequent $\Gamma \Rightarrow A$, as otherwise the conclusion of this $(\bot)$ is a principal premise of an elimination rule. Otherwise the argument will be similar to the one in Lemma 4.1.

The principal branch ends in the endsequent of the derivation and begins in an axiom $\gamma \Rightarrow \gamma$. Tracing the principal branch from the bottom up, we see that every sequent in that branch has a form (7) with empty Δ and P strictly positive in Γ. In particular, principal formulas of all $\vee^-$-inferences in the principal branch are strictly positive subformulas of Γ. Applying induction hypothesis to side premises of the last inference L in the principal branch, we get the first of our three conditions in all cases except when the third condition holds for the minor premise of the inference L. Consider both possible cases.

$$\dfrac{\Pi \Rightarrow A \to B \quad \dfrac{\Phi \to \bot}{\Phi \Rightarrow A}}{\Pi, \Phi \Rightarrow B}$$

Because the major premise $\Pi \Rightarrow A \rightarrow B$ is not introduced by $(\bot)$, formula $A \rightarrow B$ is strictly positive in Γ by induction hypothesis; hence, B is negative in Γ and so the second condition holds.

$$\dfrac{\Pi \Rightarrow A \vee B \quad \dfrac{A, \Phi \Rightarrow \bot}{A, \Phi \Rightarrow C} \quad B, \Psi \Rightarrow C}{\Pi, \Phi, \Psi \Rightarrow C}$$

Because the deduction is pruned, the second minor premise $B, \Psi \Rightarrow C$ is not introduced by $(\bot)$. By induction hypothesis C is strictly positive in Γ. ■

THEOREM 5.5
The problem of DC derivability for arbitrary propositional sequents is CoNP-complete.

The proof is the same as for Theorem 4.5. The notion of DC refutation is defined in the same way as before, only the condition

$$\bot \notin \mathcal{R}$$

is added to the clause 2. After this Lemmata 4.7, 4.8 are proved exactly as before.

6 Gentzen-style formulation

Some of the previous definitions and especially proofs can seem more transparent in terms of Gentzen-style rules instead of natural deductions. To simplify formulation, we consider a fragment without negation, i.e. without constant $\bot$ in the language.

Let's recall a translation of natural deduction due to D. Prawitz [5], which preserves normal form, i.e. transforms normal natural deductions into *cut-free* derivations.

Introduction rules of natural deduction and the rule $(\bot)$ coincide with succeedent rules of the Gentzen-style system (especially in our sequent formulation). Elimination rules of natural deduction go into antecedent rules of the Gentzen-style system applied in reverse order.

Before making this more precise, consider an example. A natural deduction is presented first, then corresponding Gentzen-style derivation.

$$\dfrac{\dfrac{p\&(p \rightarrow q) \Rightarrow p\&(p \rightarrow q)}{p\&(p \rightarrow q) \Rightarrow p \rightarrow q}\,\&^- \quad \dfrac{p\&(p \rightarrow q) \Rightarrow p\&(p \rightarrow q)}{p\&(p \rightarrow q) \Rightarrow p}\,\&^-}{p\&(p \rightarrow q) \Rightarrow q}\,\rightarrow^-$$

$$\dfrac{\dfrac{p \Rightarrow p \quad q \Rightarrow q}{p, (p \rightarrow q) \Rightarrow q}\,\rightarrow\Rightarrow}{p\&(p \rightarrow q) \Rightarrow q}\,\&^{\rightarrow}$$

Now consider arbitrary normal natural deduction ending in an elimination rule and the principal branch

$$\begin{array}{c} \Gamma \Rightarrow A_1 \\ \hline \Gamma \Rightarrow A_2 \\ \vdots \\ \Gamma \Rightarrow A_n \end{array} \tag{11}$$

of this deduction. This branch consists of principal premises of consecutive elimination rules with the very first of these sequents being an axiom, so that $A_1 \in \Gamma$ and we write Γ as A_1, Γ below. For every $i<n$, the formula A_{i+1} is the antecedent side formula of the antecedent Gentzen-style rule with the principal formula A_i.

Performing in reverse order antecedent Gentzen-style rules corresponding to elimination rules in (11) (and assuming inductively that cut-free derivations of minor premises are already constructed) we get required cut-free derivation of the endsequent where the principal branch is as follows:

$$\dfrac{\begin{array}{c} A_n,\Gamma \Rightarrow A_n \\ \vdots \\ A_2,\Gamma \Rightarrow A_n \end{array}}{A_1,\Gamma \Rightarrow A_n}$$

This construction shows in particular that a DC deduction goes into a Gentzen-style derivation by antecedent rules only.

Definition 6.1
Ant derivation is a Gentzen-style derivation by antecedent rules.

Let now revisit some of the definitions and results above.

A new proof of Theorem 3.2. By induction on the number of inferences, it is straightforward to prove that every sequent in a given $Ant_{\&,\rightarrow}$-derivation is of the form (6). No special analysis of the principal branch is needed. ■

A new proof of Theorem 4.5. An analogue of Lemma 4.1 is easily proved by induction on a derivation. No special analysis of the principal branch is needed. For Gentzen-style derivation, it is possible to define DC refutation in more 'geometric' way, as a non-closed branch of a proof search tree.

Definition 6.2
A $(\rightarrow\Rightarrow)$ inference is *simple* if its l.h.s. premise is an axiom:

$$\dfrac{A,\Gamma \Rightarrow A \quad B,\Gamma \rightarrow G}{A \rightarrow B, A, \Gamma \Rightarrow G} \rightarrow\Rightarrow$$

Definition 6.3
A set of formulas $\mathcal{R}$ is *Ant refutation* of a sequent $\Gamma \Rightarrow A$ if $\mathcal{R} \supseteq \Gamma$, $A \notin \mathcal{R}$ and $\mathcal{R}$ is closed under applications of antecedent rules $(\&\Rightarrow)$, $(\vee\Rightarrow)$ and simple applications of $(\rightarrow\Rightarrow)$, i.e.

(1) If $(B\&C) \in \mathcal{R}$ then $B \in \mathcal{R}$ and $C \in \mathcal{R}$,
(2) If $(B \vee C) \in \mathcal{R}$ then $B \in \mathcal{R}$ or $C \in \mathcal{R}$,
(3) If $B, (B \rightarrow C) \in \mathcal{R}$ then $C \in \mathcal{R}$.

With this definition Lemma 4.7 gets a new proof.

Lemma 6.4
Every Ant derivation of $\Gamma \Rightarrow A$ can be transformed into a derivation of $\Gamma \Rightarrow A$ in which all $\rightarrow\Rightarrow$ inferences are simple.

Proof. Induction on the number of non-simple $\rightarrow\Rightarrow$ inferences. Take uppermost such inference and push it up the left branch. For example,

$$\dfrac{\dfrac{C,D,\Gamma \Rightarrow A}{C\&D,\Gamma \Rightarrow A} \quad B, C\&D, \Gamma \Rightarrow G}{A \rightarrow B, C\&D, \Gamma \Rightarrow G} \quad \text{becomes} \quad \dfrac{\dfrac{C,D,\Gamma \Rightarrow A \quad B, C\&D, \Gamma \Rightarrow G}{A \rightarrow B, C\&D, C, D, \Gamma \Rightarrow G}}{A \rightarrow B, C\&D, \Gamma \Rightarrow G}$$

When an axiom is reached on the l.h.s. branch, the $\to\Rightarrow$ inference is simple. ■

Now Lemma 4.8(b) is proved as follows. Suppose $\mathcal{R}$ is an Ant refutation of $\Gamma\Rightarrow A$, but $\Gamma\Rightarrow A$ is Ant derivable. By the previous Lemma 6.4, there is a derivation in which all $\to\Rightarrow$ inferences are simple. Take the branch of this derivation determined by $\mathcal{R}$ so that every antecedent in every sequent $\Delta,\Gamma\Rightarrow A$ in this branch is contained in $\mathcal{R}$. This means that at $\vee$-branching the branch containing the side formula in $\mathcal{R}$ is taken (and there are no $\to\Rightarrow$-branchings by simplicity). Because $A\notin\mathcal{R}$, this branch does not contain an axiom.

This implies Theorem 4.5 as before.

7 Relation to other work

Although most of the automated deduction community has focused on sequent calculi, our work is by no means the first attempt at search for natural deduction proofs. One of the principal difficulties in searching directly for natural deduction proofs is that the search is mostly limited to normal deductions, as they satisfy the subformula property. W. Sieg and his colleagues have developed the *intercalation calculus* that does just this for both classical and intuitionistic logic. See [7, 8] for details. Their basic idea is to attempt to 'close the gap' between assumed formulas and goal formula 'by systematically using elimination rules "from above" and inverted introduction rules "from below"' [8, p. 170]. ADC shares this idea with the intercalation calculus. The fundamental difference between the two algorithms consists in the fact that their search procedure looks at *all possible* ways of 'closing the gap'.

To see this difference in action, let us re-examine the two examples from the introduction that showed that ADC is not complete. While we refer the reader to [7, 8] for explicit formulation of the intercalation rules, note that their rules operate on triples $\alpha;\beta?G$ where α is a set of assumptions, G is a goal formula and β is a sequence of formulas obtained from &- and $\to$-elimination of formulas in α.

Recall that ADC could not derive

$$\Rightarrow (p\&q\to r)\to(p\to(q\to r))$$

The branch in the intercalation search tree yielding the normal derivation of this sequent begins with three $\to\uparrow$ rules corresponding to $\to$ introduction rules, yielding the new question

$$p\&q\to r,p,q;\emptyset?r$$

It was here that ADC failed because it cannot introduce the conjunction in the antecedent in the implication. In the intercalation calculus, however, the $\to\downarrow$ rule applies to generate two sub-questions $p\&q\to r,p,q;\emptyset?p\&q$ and $p\&q\to r,p,q;r?r$. The branch for the second question is immediately closed, as $r\in\alpha\beta$; the first question contains one more branching for a $\&\uparrow$ rule.

Similarly, for $p\vee q\Rightarrow q\vee p$, the intercalation calculus allows the disjunction in $q\vee p$ to be introduced before a $\vee^-$ rule has been applied. By forbidding introduction rules to precede elimination rules, we lose completeness but avoid PSPACE complexity.

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