DETERMINATION OF COEFFICIENTS OF INFILTRATION EQUATIONS

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ABSTRACT

The paper describes a procedure which may be used to evaluate the coefficients of different infiltration equations from data obtained on surface irrigation systems. Measurements are required of: the discharge rate, the rate of advance and the depths of water on the field at different points and times during the advance. The calculation procedure followed in evaluating the coefficients employs the standard least-squares, curve fitting technique.

A comparison was made between the depths of infiltration along a border dyke, as calculated by the equations of Horton, Kostiakov and Philip, with changes in soil moisture measured at different points along the strip with a neutron probe. The results of this study showed that the average "volumetric" depths-of-infiltration along the strip calculated by the three methods were in close agreement with the depth estimated from the soil moisture changes; the differences ranged from 4 to 9 percent of the measured amount. Wide differences were noted however between "measured" soil moisture changes and "calculated" depths-of-infiltration at specific locations along the strip.

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INTRODUCTION

Because of the many factors which affect the infiltration rate of a natural soil under field conditions (e.g., soil moisture, soil anisotrophy, and others) it has been proposed that the coefficients of the infiltration equation used in the design of a surface irrigation system should be based on data obtained under field conditions (Gray and Ahmed, 1965; Christiansen et al, 1966; Norum and Gray, 1970). Christiansen et al (1966) developed a method for determining the coefficients of the infiltration equation of the Kostiakov type (1932) from advance data. In the derivation it was assumed that the advance of water down a border strip follows a power function with time and that the average depth of surface water is constant with time. Norum and Gray (1970) employing Philip and Farrell's solution (1964) of Lewis and Milne's equation (1938), developed a curve matching technique to determine the infiltration equations presented by Kostiakov (1932) and Philip (1964). As Philip and Farrell's solution (1964) is based on the assumption that the average surface depth of water remains constant with time, the accuracy of the technique proposed by Norum and Gray (1970) is limited when applied to field conditions as the average surface depth generally changes with time (Farrell, 1963; Fok and Bishop, 1965; Lin, 1970).

In this paper, a method is proposed which utilizes statistical techniques to evaluate the coefficients of the different infiltration equations from advance data. With the method no restrictions are imposed on the manner in which the surface depth varies with time.

GENERALIZED CONSERVATION EQUATION

When a given discharge is introduced to the upper end of a sloping field (as illustrated in Figure 1), the conservation equation expressing the flow per unit width can be written as:

$$\int_o^t q dt = V + \int_o^X z(t - \tau) dx \qquad \ldots 1$$

where q = the discharge or input rate per unit width,

t = the time for the flow to reach a distance, X,

V = the volume of water stored on the surface, a function of time, taken equal to the product DX in which; D is the average surface flow depth and X is the distance of advance of the wet front, a function of time,

z = the cumulative depth of infiltration, a function of the infiltration opportunity time, $t - \tau$,

τ = the time for the flow to reach a distance, x, $(o < \tau < t)$, and

x = the distance to any point along the field, $(o < x < X)$.

Equation 1 can be written in a different form using the boundary conditions: $x(t - \tau) = o$, when $x = X$, as

$$Q - V = \int_o^t I(t - \tau)\ x(\tau)\ d\tau \qquad \ldots 2$$

where $Q = \int_o^t q dt$, the cumulative discharge per unit width,

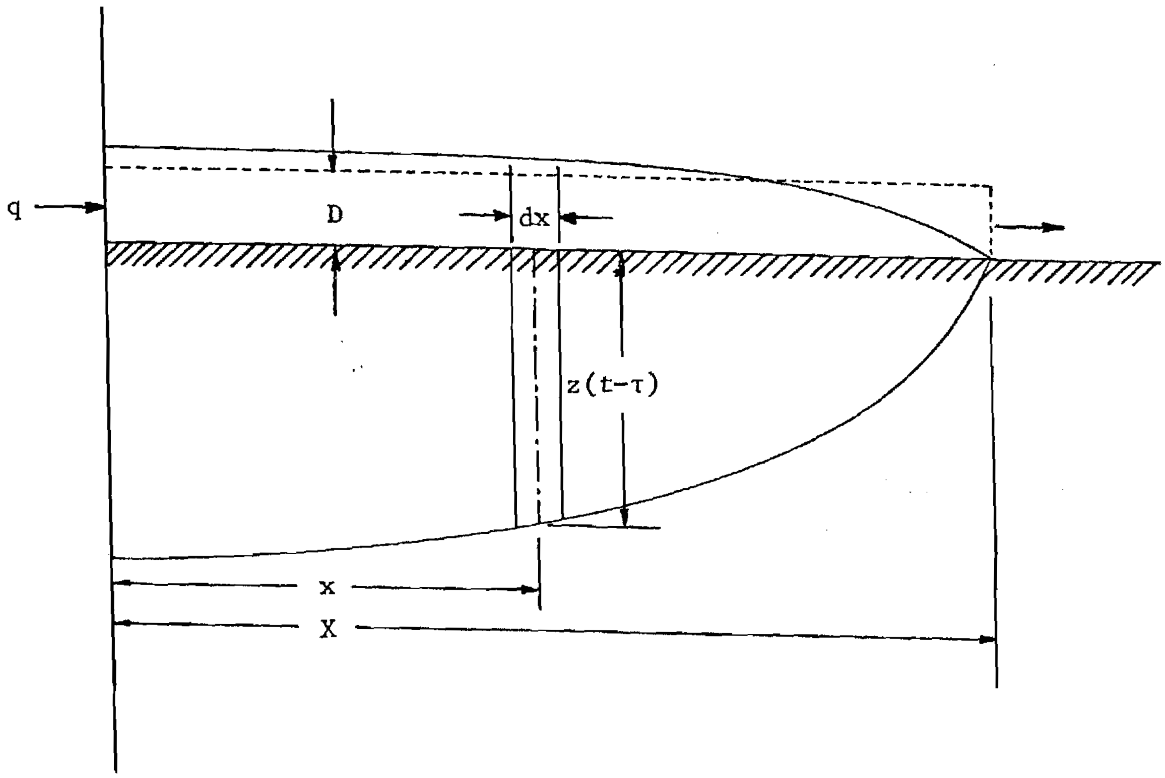

Figure 1. Overland Flow Down a Sloping Field

$I(t-\tau) = \frac{dz(t-\tau)}{d(t-\tau)}$, the infiltration rate, a function of opportunity time, $t - \tau$, and,

$Q-V$ = the volume of water infiltrated, a function of time whose magnitude is obtained experimentally from measurements of the input rate, advance, surface depth profiles and time.

Taking the LaPlace Transform of Equation 2 and simplifying the result one obtains:

$$I(t) = L^{-1}\left[\frac{L\{M(t)\}}{L\{x(t)\}}\right] \qquad . . . 3$$

in which L = LaPlace transform,

$M(t) = Q - V$, and

$x(t) = X$.

Theoretically, it would appear that I(t) could be obtained from Equation 3 even if M(t) and x(t) were in numeric rather than analytic form. However, from a practical standpoint, obtaining an inverse transform from a numerically-specified transform is inherently inaccurate.

It has been well established from field experiments as those reported by Gray and Ahmed (1965) and Christiansen, et al (1966) and from laboratory results presented by Lin (1970) that for a constant input rate the length of advance can be approximated by a power function of time as,

$$X = ht^{\beta} \qquad . . . 4$$

where h and β are coefficients.

Substituting Equation 4 into Equation 3 and taking the inverse transform it follows that

$$M(t) = h\Gamma(\beta+1)L^{-1}\left[\frac{L\{I(t)\}}{s^{\beta+1}}\right] \quad \ldots 5$$

where Γ = the gamma function, and

s = the parameter in the LaPlace transform.

M(t) cannot in general be described by an analytic function. However, if it is assumed that I(t) can be approximated by some analytic function of t with specified coefficient; C_1, C_2 ..... C_n, then Equation 5 can be written as,

$$\hat{M}(t) = F(t, C_1, C_2 \ldots\ldots C_n) \quad \ldots 6$$

where $\hat{M}(t)$ is the calculated infiltration volume. Clearly, therefore, discrete values of M(t), as those obtained in field irrigation tests, can be operated on by a least-square fitting technique to determine the best fit values of the coefficients C_1, C_2 ..... C_n.

Equation 5 can be written in a form more compatible for use in analyzing field data as:

$$Q - V = h\Gamma(\beta+1)L^{-1}\left[\frac{L\{I(t)\}}{s^{\beta+1}}\right] . \quad \ldots 7$$

CONSERVATION EQUATIONS BASED ON DIFFERENT INFILTRATION EQUATIONS

The coefficients of any cumulative infiltration equation can be obtained by regressing Equation 7. To illustrate the procedure, suppose the cumulative infiltration varies with time as suggested by Philip (1964), that is,

$$z(t) = St^{\frac{1}{2}} + At \quad \ldots 8$$

where S and A are constants.

The infiltration rate, I(t) is therefore,

$$I(t) = \frac{S}{2} t^{-\frac{1}{2}} + A \qquad . . . 9$$

and the LaPlace transform of Equation 9 is:

$$L\{I(t)\} = \frac{S\sqrt{\pi}}{2} \frac{1}{s^{\frac{1}{2}}} + \frac{A}{s} \qquad . . . 10$$

Substituting Equation 10 into Equation 7 and simplifying, one obtains;

$$Q\text{-}DX = ht^{\beta} \left[\frac{S\sqrt{\pi}}{2} \frac{\Gamma(\beta+1)}{\Gamma(\beta+3/2)} t^{\frac{1}{2}} + \frac{A}{\beta+1} t\right] \qquad . . . 11$$

Since $X = ht^{\beta}$, the final form of the conservation equation is:

$$\frac{Q}{X} - D = \frac{S\sqrt{\pi}}{2} \frac{\Gamma(\beta+1)}{\Gamma(\beta+3/2)} t^{\frac{1}{2}} + \frac{A}{\beta+1} t . \qquad . . . 12$$

Obviously the form of equation is a simple polynomial function between, $\frac{Q}{X} - D$, and time, t.

The conservation equations developed in a similar manner for different cumulative infiltration functions and their corresponding regression equations are given in Table I. For most cases, standard programs available on most digital computers can be used to obtain the "least-squares" estimators of the coefficients from field data. Additional details concerning the computational procedure are given by Chow (1964).

RESULTS AND DISCUSSION

During 1973 a series of tests were conducted on several border strips (15.24 m x 289.6 m) located in the South Saskatchewan Irrigation Project. In each test the following measurements were taken: (a) soil moisture profiles at different points along the centerline of the strip immediately preceding and following irrigation, (b) the discharge rate, (c) the rate of advance, (d) the surface depths of water at different points and times during the advance, and (e) the rate of recession.

A representative set of data obtained from the field experiments is given in Table II. The least-squares estimator of the coefficient, β, (Equation 4) was obtained by a standard regression procedure. The "least-squares" coefficients of three standard cumulative infiltration equations (Horton, Kostiakov and Philip) calculated from these data are given in Table III.

According to the continuity equation (Equation 7) the quantity, Q - DX, should be equal to the volume of infiltration. Values of this quantity obtained from measurements of inflow rate and the rate and depth of advance expressed as an average volume per unit length are compared with the volumes of infiltration calculated using Equations 13, 16 and 19 (Table I) and the results plotted as a cumulative curve in Figure 2. It is evident from the figure that;

1. The cumulative average values of the quantity, Q - DX, and the cumulative "calculated" infiltration amounts are in close agreement. This result would be expected as the same field data are used to evaluate the quantity; Q - DX, and the coefficients of the infiltration equations. As such, therefore, the results cannot be used as a direct test of the "goodness" of the different infiltration equations. Nevertheless, they indicate that:

   (a) The least-squares parameters of the infiltration equations as determined from analysis of data collected over relatively long "lengths-of-run" may be used to calculate the average infiltration amounts with time and distance. It should be recognized that with the least-squares method the conservation of mass does not strictly apply. The method simply gives the best fit estimators of an equation over the range of data used.

(b) The rate of advance may be described by a power function of time.

(c) Only small differences exist between the quantities of infiltration calculated by the three different infiltration equations. As the power equation gives comparable results to those obtained by the other functions it is assumed that because of its simplicity, this form of infiltration equation would have the wider practical use.

The preceding discussion has stressed that as the same field data were used to evaluate the term; $Q - DX$, and the coefficients of the infiltration functions, $z(t)$, direct comparison of the "measured" and "calculated" infiltration amounts are biased. As an independent test of the method, the cumulative infiltration amounts which occurred during the irrigation as determined from measurements of the soil moisture changes within a 1.22 m profile with a neutron probe at selected points in the field were compared with the cumulative depths of infiltration at the same points calculated by integrating the different infiltration functions over the opportunity time (the difference in time between the advance and recession curves).

Figure 3 shows a representative set of results obtained by these calculations. As shown in the figure wide differences between "measured" and "calculated" infiltration occur at different points along the strip.

Such differences may be expected because: (a) the "measured" infiltration amounts represent point estimates which may vary greatly in space whereas the "calculated" infiltration values are based on average estimates of the infiltration characteristics over the length of advance, and (b) significant errors may have occurred in the measurement of different variables; particularly in the measurements of the changes in soil moisture at the surface depths. Comparing, on a volumetric basis, the "measured" and "calculated" amounts of infiltration over the length of advance (195 m), assuming a linear variation in the depth of infiltration between sampling points, it was found that the cumulative amounts calculated by the Horton (Equation 13), Kostiakov (Equation 16) and Philip (Equation 19) equations differ from the measured infiltration by 4.3%, 6.8% and 9.0% respectively. These percentages represent the ratio of the differences to the measured values. It is likely that the magnitudes of these differences are within the error of measurement of the "measured values". Hence the results should be used with high confidence for comparing the accuracy of the different infiltration functions for estimating the actual amounts of infiltration. Based on these considerations and the fact that the estimates of infiltration given by the different equations are approximately equal - the choice of a given equation to describe the infiltration phenomenon would likely be based on practical considerations.

SUMMARY AND CONCLUSIONS

The paper oultines a procedure which may be used to evaluate the parameters of different infiltration equations from data collected on surface irrigation systems. The method employs standard statistical curve fitting techniques using measurements of the rate of advance and the depth of surface flow at different times at selected stations during the advance. Comparisons of the average infiltration amounts over a border strip as calculated by three different infiltration equations compared favourably with measured amounts.

The particular advantages of the procedure are that: (a) it is simple, direct and easy to apply as the programs used are generally available on most digital computers, and (b) it requires only field measurements of the surface flow phenomena which can readily be obtained without sophisticated instrumentation. The method is limited however in that the coefficients are the best fit estimates of the infiltration equation for the entire range of field data used in their derivation.

REFERENCES

CHOW, V.T., 1964. Handbook of applied hydrology. McGraw-Hill Book Co. New York.

CHRISTIANSEN, J.E., A.A. BISHOP, F.W. KIEFER Jr., and YU-SI FOK, 1966. Evaluation of intake rate constants as related to advance of water in surface irrigation. Trans. Amer. Soc. Agri. Eng., Vol.9, No.5-1, pp.671-674.

FARRELL, D.A., 1963. The hydraulics of surface irrigation. Unpublished Ph.D. Thesis, University of Melbourne, Australia.

FOK, Y.S., and A.A. BISHOP, 1965. Analysis of water advance in surface irrigation. J. Irrigation and Drainage Div. Amer. Soc. Civil Eng., Vol.91(IR1), pp.99-116.

GRAY, D.M. and M. AHMED, 1965. Rational approach applied to the design of border dyke systems. Can. Agric. Eng., Vol.7, No.1, pp.30-33,44.

HORTON, R.E., 1940. An approach toward a physical interpretation of infiltration - capacity. Soil Sci. Soc. Amer. Proc. 5:399-417.

KOSTIAKOV, A.N., 1932. On the dynamics of the coefficients of water percolation in soils and of the necessity of studying it from a dynamic point of view for purposes of amelioration. Trans., 6th Committee Inter. Soc. Soil Sci., pp.17-21.

LEWIS, M.R. and W.E. MILNE, 1938. Analysis of border irrigation. Agric. Eng., Vol.19, No.6, pp.267-272.

LIN, W., 1970. Hydrodynamics and kinematics of overland flow using a laminar model. Unpublished Ph.D. Thesis, University of Saskatchewan, Saskatoon, Saskatchewan, Canada, pp.1-117.

NORUM, D.I. and D.M. GRAY, 1970. Infiltration equations from rate-of-advance data. J. Irrigation and Drainage Div. Amer. Soc. Civil Eng., Vol.96(IR2), pp.111-119.

PHILIP, J.R., 1964. An infiltration equation with physical significance. Soil Sci. 77:153-157.

TABLE II. EXPERIMENTAL DATA

t (min)	X	Q (m^3/m)	D (cm)	Q/X-D (cm)
20	30.48	1.71	4.44	1.17
40	55.47	3.96	5.49	1.66
60	79.25	6.33	5.64	2.34
80	99.06	8.69	5.54	3.23
100	115.82	11.05	5.69	3.85
120	131.97	13.52	6.15	4.10
140	152.4	16.11	6.02	4.55
160	177.39	18.69	6.25	4.82
180	195.07	21.27	5.46	5.44

$\beta = 0.7667$

TABLE III COEFFICIENTS OF DIFFERENT INFILTRATION EQUATIONS

Type	Cumulative Depths [a]	
Horton (1940)	$z = 0.0347t + 2.027\ (1-e^{-0.0437t})$	... 27
Kostiakov (1932)	$z = 0.198t^{0.725}$	... 28
Philip (1964)	$z = 0.280t^{\frac{1}{2}} + 0.0273$	... 29

[a] z = cumulative depth of infiltration in cm,
t = time in min

TABLE 1. CONSERVATION EQUATIONS AND THE CORRESPONDING REGRESSION EQUATIONS FOR DIFF
A, B, S, k, r, α, are coefficients to be determined from experimental data;
In the regression equations, x, y and β are variables, and a, b and c are t

CUMULATIVE INFILTRATION	CONSERVATION	GENERA REGRESSI
Horton (1940) $z(t) = B(1-e^{-rt}) + At$ (13)	$\frac{Q}{X} - D = rB\,\Gamma(\beta+1)\left[\sum_{n=o}^{\infty} \frac{(-r)^n\, t^{n+1}}{\Gamma(\beta+n+2)}\right] + \left(\frac{A}{\beta+1}\right)t,$ $\frac{Q}{X} - D \approx B\left(1-\frac{\beta}{rt}\right) + \left(\frac{A}{\beta+1}\right)t,$ $t >> 1.$ (14)	y = a +
Kostiakov (1932) $z(t) = kt^{\alpha}$ (16)	$\frac{Q}{X} - D = \left[\frac{\Gamma(\alpha+1)\cdot\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\cdot k\right]t^{\alpha}.$ (17)	y = a +
Philip (1964) $z(t) = St^{\frac{1}{2}} + At$ (19)	$\frac{Q}{X} - D = \frac{S\sqrt{\pi}}{2}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+3/2)}\right]t^{\frac{1}{2}} + \left(\frac{A}{\beta+1}\right)t.$ (20)	y = ax +
Lewis and Milne (1938) $z(t) = B(1-e^{-rt})$ (22)	$\frac{Q}{X} - D = rB\,\Gamma(\beta+1)\left[\sum_{n=o}^{\infty} \frac{(-r)^n\, t^{n+1}}{\Gamma(\beta+n+2)}\right],$ $\frac{Q}{X} - D \approx B\left(1-\frac{\beta}{rt}\right),$ $t >> 1.$ (23)	y = a +
Christiansen et al. (1964) $z(t) = kt^{\frac{n}{m}} + At,$ (m>n>o, are known integers) (25)	$\frac{Q}{X} - D = \left[\frac{\Gamma\left(\frac{n}{m}+1\right)\cdot\Gamma(\beta+1)}{\Gamma\left(\frac{n}{m}+\beta+1\right)}\cdot k\right]t^{\frac{n}{m}} + \left(\frac{A}{\beta+1}\right)t.$ (26)	$y = ax^{n}$

NG REGRESSION EQUATIONS FOR DIFFERENT CUMULATIVE INFILTRATION EQUATIONS.
etermined from experimental data; e is the base of the natural logarithms.
e variables, and a, b and c are the constants for "best fit" lines.

ON	GENERAL FORM OF REGRESSION EQUATION	REGRESSION COEFFICIENTS
$\left.\frac{r)^{n}\ t^{n+1}}{\Gamma(\beta+n+2)}\right] + \left(\frac{A}{\beta+1}\right)t,$ $\left.\right)t,$ (14)	$y = a + bx + cx^2$ (15)	$y = \left(\frac{Q}{X} - D\right)\ t^{\gamma};\ x = t;$ $a = -\frac{B\beta}{r};\ b = B;\ c = \frac{A}{\beta+1}$.
$\left.k\right]t^{\alpha}$. (17)	$y = a + bx$ (18)	$y = \ell n\ \left(\frac{Q}{X} - D\right);\ \ x = \ell n\ t;$ $a = \ell n\ \left[\frac{\Gamma(\alpha+1)\cdot\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\cdot k\right];\ b = \alpha$.
$+ \left(\frac{A}{\beta+1}\right)t$. (20)	$y = ax + bx^2$ (21)	$y = \frac{Q}{X} - D;\ x = t^{\frac{1}{2}};$ $a = \frac{S\sqrt{\pi}}{2}\ \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+3/2)}\right];\ b = \frac{A}{\beta+1}$.
$\left.\frac{r)^{n}\ t^{n+1}}{\Gamma(\beta+n+2)}\right],$ (23)	$y = a + bx$ (24)	$y = \frac{Q}{X} - D;\ \ x = \frac{1}{t};$ $a = B;\ \ b = -\frac{B\beta}{r}$.
$\left.\right]\ t^{\frac{n}{m}} + \left(\frac{A}{\beta+1}\right)t.$ (26)	$y = ax^n + bx^m$ (27)	$y = \frac{Q}{X} - D;\ x = t^{\frac{1}{m}};$ $a = \left[\frac{\Gamma\left(\frac{n}{m}+1\right)\cdot\Gamma(\beta+1)}{\Gamma\left(\frac{n}{m}+\beta+1\right)}\cdot k\right];\ b = \frac{A}{\beta+1}$.